THE ELASTIC STRENGTH OF GUNS

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BY

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PREFACE

This little book was prepared primarily for use by the midshipmen at the U. S. Naval Academy. It essays to present the subject of the elastic strength of guns as concisely as is consistent with clearness, and to that end treats only of steel guns of modern construction, built-up or wire-wound.

The hypothesis that permanent set will not occur unless the resultant *strain* in some direction exceeds the limit of elastic strain, regardless of what the stresses may be, is adopted. This hypothesis appears to the writer to be the only reasonable one, but it is to be regretted that its truth has never been demonstrated experimentally.

The longitudinal stress is taken to be zero, an assumption made by Claverino in his first treatise on the "Resistance of Hollow Cylinders," published in the "Giornale d'Artiglieria" in 1876, and adopted by Birnie in his exhaustive studies of the resistance and shrinkages of built-up cannon.

The formulæ for wire-wound guns were originally deduced by the writer some twenty years ago, and were then first published in the *U. S. Naval Institute Proceedings*.

A number of illustrative examples are solved in the text, and others, with their answers, follow each chapter.

U. S. Naval Academy,
Department of Mechanics,
November, 1904.

CONTENTS

ADDENDA.

Page 18.—If the cylinder is cut by a diametral plane, the whole pressure acting outward upon the section is $2P_0R_0$, and the whole pressure acting inward upon the section is $2P_nR_n$, so that the total force tending to burst the cylinder is $2P_0R_0 - 2P_nR_n$. This force must be balanced by the total stress developed in the two sections of the cylinder walls, each of which is $\int_{R_0}^{R_n} t\,dr$.

Page 19. $2p+k=-r\frac{dp}{dr}$; $\int\frac{dp}{2p+k}=-\int\frac{dr}{r}$;

$$\tfrac{1}{2}\log(2p+k)=\log\frac{1}{r}+\log k_1;$$

$$\sqrt{2p+k}=\frac{k_1}{r};\quad 2p+k=\frac{k^2{}_1}{r^2}.$$

Page 24.—Since e_p is the radial strain, $e_p \times dr$ is the change of length of dr, and so the whole change of thickness of the cylinder wall is $\int_{R_0}^{R_n} e_p dr$.

THE ELASTIC STRENGTH OF GUNS

CHAPTER I.

INTRODUCTORY.

1. **Stress and Strain.**—We give the name *stress* to a mutual action between the parts of a body, or between one body and another, causing, or tending to cause them to move relative to one another; it is any pair of equal and opposite actions each of which is what is called a force.

Thus, if a rope be stretched vertically downwards from A to B, we speak of the tension T of the rope as the force T acting downward on A, or as the force T acting upward on B, according as we are considering A or B; but we speak of the action in the rope, which tends to break it, as the stress in the rope.

2. We call the change of volume or figure of any solid or liquid under the action of force a *strain*.

Thus, if a bar is lengthened or shortened, it is strained; a compressed liquid is strained; a stone, a piece of metal, or other part of any structure, is said to experience a strain if it be bent, or twisted, or compressed, or dilated, or in any manner distorted. Furthermore, any change in the configuration of a group of bodies whose relative positions are subject to fixed conditions is called a strain. Thus, any structure is said to strain when its different parts experience relative motion, as, for example, a ship "strains" in a seaway.

3. If we imagine any plane area within a strained body as forming a division between the parts of the body on either side of it, then the force which each of the two parts exerts upon the other is one of the pair of forces which constitute the stress on the area. In other words, the stress on any sectional area is the pair of equal and opposite actions which hold the area in its state of strain.

4. *The intensity of stress* is the number of units of force per unit of area. We shall always express it in tons weight, or pounds

weight, per square inch; and, for brevity, we shall use the word stress as meaning "intensity of stress," always applying the term "total stress" to the whole force acting on any area. If the intensity of the stress (p) is the same at all points of a given area (A), the stress on the area is said to be uniformly distributed, and P being the total stress on the area, we have $p = \frac{P}{A}$. If the stress is not uniformly distributed, its intensity at any point is given by $p = \frac{dP}{dA}$, where dP is the total stress on the elementary area dA.

5. Hook's Law.—Every stress is accompanied by a strain, and experiments show that in all solid bodies the strain is proportional to the stress which causes it, provided the stress does not exceed certain limits which vary with the material. This is what is known as *Hook's law,—"ut tensio sic vis"* (*as the extension so the force*).

6. The simplest form of stress is that which exists in a bar of uniform section to which equal and opposite forces are applied axially, tending to lengthen or shorten it. If the forces act to lengthen the bar, the stress is called tension, and if they act to shorten it, the stress is called compression; but mathematically considered compression is merely negative tension.

The strains accompanying tension are an elongation in the direction of the pull and a contraction in all directions perpendicular to it; while the strains accompanying compression are the reverse, *i. e.,* a shortening in the direction of the push and an expansion in all directions perpendicular to it. These strains are elastic, that is, they disappear with the removal of the forces which caused them, so long as the tension—or the compression, as the case may be—does not exceed a value which is called the *elastic limit* of the material. Within that limit the strains follow Hook's law.

7. If P be the total pull (or push) on the bar, and A be the area of its right section, the total stress on any such section is P, and, since it is uniformly distributed, its intensity is $p = \frac{P}{A}$. The *elastic limit* * is the value of p beyond which the strain ceases to

* Some writers use the term elastic limit to denote the greatest elastic *strain* under simple tension or compression, instead of the greatest *stress* causing only elastic strains. We shall use the term *elastic limit of strain* to distinguish the former concept, and shall use *elastic limit* to denote the *elastic limit of stress.*

be wholly elastic; if this value is exceeded, the bar takes a permanent set, *i. e.,* when released it will be found to be longer (or shorter) than it was originally. With some materials, notably cast iron, the elastic limit under compression considerably exceeds that under tension, but in the case of steel the difference, if it exists, is not important. The elastic limit of the steel forgings used in modern gun construction is from 35,000 to 75,000 pounds per square inch.

8. **The Modulus of Elasticity.**—Within the elastic limit the ratio of stress to strain is, by Hook's law, a constant, and the value of this constant for the case of simple tension or compression is called the *modulus of elasticity* and is denoted by E. That is to say, if e is the change of length per unit length under the stress $p = \frac{P}{A}$, then $E = \frac{p}{e}$.

Since e is the relative, not the total, strain, it is an abstract number, being, in the case considered, the total change of length of the bar (due to its tension or compression) divided by its length when free. Consequently E is a quantity of the same kind as p and its value depends upon the units in which p is expressed.

When p is given in pounds per square inch, E has the value 29,000,000 for steel; when p is expressed in tons per square inch, E has the value 13,000.

Evidently E is the stress which would double the length of a bar under tension (if it continued to obey Hook's law to that point), since when $e = 1$, $p = E$.

It must be understood that E is the value of the stress on a right section of the bar divided by the strain perpendicular to that section, or in the direction of the external forces causing the strain; the strains at right angles to the axis of the bar, though proportional to the principal strain, are less in value, their ratio to it, determined by experiment, being, in this work, taken to have the value $\frac{1}{3}$.*

9. **Example.**—As an example, suppose a round steel bar, 2 inches in diameter and 20 inches long, to be under a tension of 60 tons; then the stress on a right section of the bar is $p = \frac{60}{\pi} = 19.1$ tons

* This quantity is known as "Poisson's ratio" from the great French mathematician. Its value varies for different materials, and for steel has been taken by different authorities as ½, ⅓ and ¼. The best modern experiments assign to it a value in the neighborhood of ⅓.

per square inch; the strain in the direction of the axis of the bar is $\frac{p}{E} = \frac{19.1}{13000} = .00147$; and the strain at right angles to the axis is $\frac{.00147}{3} = .00049$. The length of the bar is increased by the tension $20 \times .00147 = .0294$ inches, making its strained length 20.0294 inches; and its diameter is diminished $2 \times .00049 = .00098$, making its strained diameter 1.99902 inches.

If the force of 60 tons were applied to compress the same bar, it would be shortened .0294 inches and its diameter would be increased .00098 inches.

Under tension the volume of the bar is increased in the ratio 1 to 1.000488; while under compression its volume is diminished in the same ratio.

10. If more than one pair of equal and opposite forces act upon a body, the stress upon any sectional area of the body is the resultant of the stresses which would be caused by the pairs of forces acting separately; and the strain at any point due to the simultaneous action of all the stresses is obtained by simply superposing the strains due to the different stresses taken separately.

Thus, taking a rectangular right prism with equal and opposite forces acting normally upon each pair of its opposite faces, let X, Y and Z be the forces acting per unit area of the respective faces: then the stress on each right section perpendicular to the X axis will be X, the stress on each right section perpendicular to the Y axis will be Y, and the stress on each right section perpendicular to the Z axis will be Z. Also, at each point in the prism, the resulting strains in the directions of the axes will be:

$$\left.\begin{aligned} e_x &= \frac{1}{E}\left(X - \frac{Y}{3} - \frac{Z}{3}\right) \\ e_y &= \frac{1}{E}\left(Y - \frac{Z}{3} - \frac{X}{3}\right) \\ e_z &= \frac{1}{E}\left(Z - \frac{X}{3} - \frac{Y}{3}\right) \end{aligned}\right\} \quad (1)$$

In these expressions e_x, e_y and e_z are the changes of length per unit length in the directions of the X, Y and Z axes, respectively, and are plus when they are lengthenings and minus when they are shortenings, provided the stresses X, Y and Z are given plus signs when they are tensions and minus signs when they are compressions.

11. Evidently if either Y or Z be of opposite sign to X, the strain in the X direction may be greater than $\frac{X}{E}$, and similarly the strains in the Y or Z directions may either be greater or less than $\frac{Y}{E}$ and $\frac{Z}{E}$ respectively, according as X, Y and Z are unlike or like forces. If, for example, $X = 15$ tons per square inch tension, and Y and Z are each 15 tons per square inch compression, we have $e_x = \frac{1}{E}\left(15+\frac{15}{3}+\frac{15}{3}\right)=\frac{25}{13000}=.001923$, and the prism would lengthen .001923 inches per inch of its free length instead of only $\frac{15}{E}=.001154$ inches per inch, as would be the case if the stress X alone acted.

12. In our investigations of the strength of guns we accept the following principle:

The total strain in any direction due to all the stresses is the measure of the tendency to yield in that direction, so that the limit of elastic strength is reached, not when the stress in any direction equals the elastic limit of the material, but when the strain in any direction equals the strain which would be caused by the direct action of a single stress equal to that elastic limit.

If, for example, a steel forging has an elastic limit of 58,000 pounds per square inch, *i. e.*, if 58,000 pounds per square inch is the greatest simple tensile stress which the steel will withstand without permanent lengthening, then for the safe use of such a forging it is necessary, and sufficient, that at no point within it shall the strain at any time exceed $\frac{58000}{E}=\frac{58000}{29000000}=.002$ inches per inch in any direction.

13. At any point in a strained solid there are always three planes, at right angles to one another, upon each of which the stress is wholly normal. These three simple stresses (tensions or compressions) are called the *principal stresses* at the point, and their directions are called the *principal axes of stress.*

In the case we are about to investigate—a hollow cylinder under internal and external fluid pressure—the principal axes of stress are evidently radial, circumferential, and longitudinal (parallel to the cylinder's axis), and the principal stresses, which we denote by p, t and q, are illustrated in Figure 1, where one of the elementary prisms of which we imagine the cylinder to be composed is shown in equilibrium under their joint action.

The strains in the directions of the principal axes of stress are called the *principal strains;* they are simple longitudinal strains (lengthenings or shortenings), and their relations to the principal stresses are those given by equations (1).

14. Since the external pressures with which we are to deal are compressive forces, it will be convenient to call the radial stress (p) plus when it acts to compress the material of the cylinder,

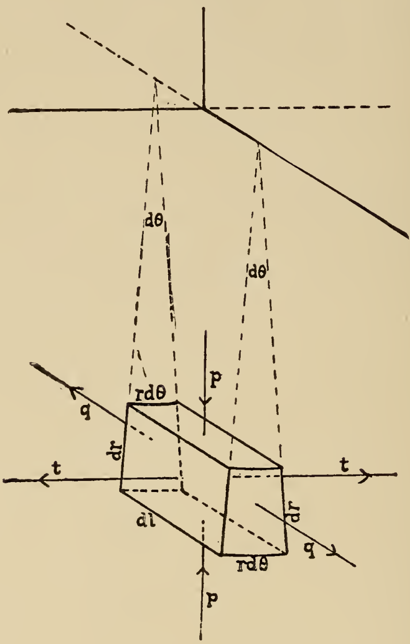


FIG. 1.

though continuing to call the circumferential stress (t) and the longitudinal stress (q) plus when they produce tension. With this convention, equations (1) become:

$$\left.\begin{aligned} e_t &= \frac{1}{E}\left[t + \frac{p}{3} - \frac{q}{3}\right] \\ e_p &= -\frac{1}{E}\left[p + \frac{q}{3} + \frac{t}{3}\right] \\ e_q &= \frac{1}{E}\left[q - \frac{t}{3} + \frac{p}{3}\right] \end{aligned}\right\} \quad (2)$$

in which e_t is the strain in the direction of the circumference, e_p the strain in the direction of the radius, and e_q the strain in the direction of the axis of the cylinder, in each case a plus value indicating extension and a minus value compression.

In the theory of elasticity it is shown that if an ellipsoid be constructed with semi-axes representing the principal stresses at a point, the stress upon any plane at the point is represented in magnitude and direction by a radius vector of the ellipsoid, which is called the ellipsoid of stress. Evidently, then, one of the three principal stresses acting at each point in a strained solid is the greatest stress at the point. In a similar way it is shown that one of the three principal strains at a point is the greatest strain at the point.

EXAMPLES I.

(1) A round steel rod 1 inch in diameter and 6 feet long is found to stretch .07 inches under a load of 10 tons. What is the intensity of the stress on its transverse section, and what is the value of the modulus of elasticity?

12.73 tons per sq. in.; 13,096 tons.

(2) What length of uniform steel rod, hanging vertically, will just carry its own weight, if the maximum allowable stress is 8 tons per square inch (steel weighs .283 lb. per cu. in.)? 5277 ft.

(3) The ends of a steel I beam whose flanges are 8 inches wide rest on stone supports. If each support takes half the total load of 20 tons, what should the length of bearing surface be, the safe compression stress for stone being 300 lbs. per square inch?

9.3 in.

(4) A bar of steel 2 inches in diameter is bent so that its axis forms the arc of a circle of 372 ft. diameter. What is the greatest strain at any point of the transverse section, and what is the greatest stress? (E for steel is 29,000,000 lbs. in.)

.000448; 12,992 lbs. per sq. in.

(5) A steel bar, 10 inches long and of square section, 1 inch on the side when free, is under 40,000 pounds tension. What are its dimensions under this stress, which is within the elastic limit?

$10.0138 \times \overline{.99954}^2$.

(6) A copper rod of square cross section, 2 inches on the side, and 5 feet long, stretches .0375 inches under a load of 40,000

pounds. What is the modulus of elasticity, and what is the cross-section while the bar is under this stress?

16,000,000 lbs. in.; 3.9983.

(7) A one-inch square steel bar of 32,000 lbs. elastic limit is under a tension of 24,000 lbs.; what pressure per square inch on all of its sides will cause it to lengthen permanently? 12,000 lbs.

(8) If a cube be subjected to equal tensions, or compressions, in each of the three directions normal to its opposite pairs of faces, what relation must exist between the stress of tension, or compression, and the elastic limit of the material in order that the cube may be permanently strained? $p = 3\theta$.

(9) The modulus of elasticity of copper being 16,000,000 (lbs. in.), how much will the length and diameter of a round copper rod, 20 inches long and 3 inches in diameter when free, change under a tensile stress of 9000 lbs. per sq. in.?

.01125 in.; .00056 in.

(10) In order to bring to the vertical opposite walls which have fallen away from each other, round steel rods of 1 in. diameter are stretched from wall to wall and after being heated to 400° C. are set up taut. What pull will each rod exert when its temperature has fallen to 200° C., supposing the walls not to have yielded at all? The coefficient of expansion of steel is .000011 for 1° C.

50,100 lbs.

(11) How much would the steel rod of Example (2), which is 5277 ft. long when free, be increased in length by the stress due to its own weight? 19.56 in.

CHAPTER II.

STRESS AND STRAIN IN SIMPLE HOLLOW CYLINDERS.

15. Consider a horizontal hollow cylinder, open at the ends, which are faced off in planes normal to the axis; and let this cylinder be filled with a fluid which is forced inward by two expanding plungers, the result being a uniform normal pressure upon the entire internal surface of the cylinder. Also let the entire outer cylindrical surface be subjected to a fluid pressure. Then, the ends of the cylinder being free, and there being no longitudinal stress upon its walls, it

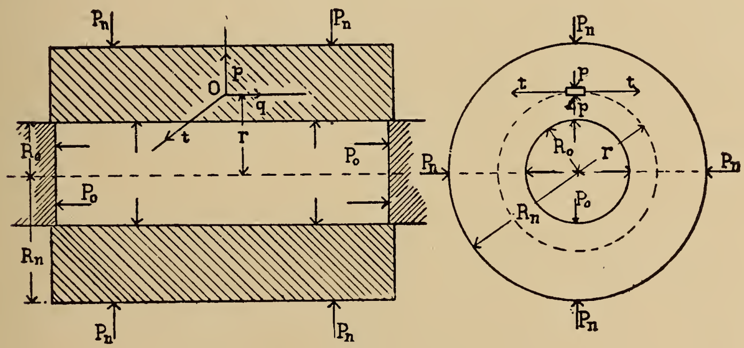


FIG. 2.

is clear that the cylinder will remain a cylinder under the action of the pressures, and that each transverse section normal to the axis will remain a plane normal to the axis. Whatever shortening or lengthening of the cylinder may result from applying internal and external fluid pressure to it must be uniform over its whole cross-section; *i. e.*, the longitudinal strain must, under the stated conditions, be constant throughout the cylindrical walls.

16. Let O be any point (of radius r) within the walls of a cylinder (Figure 2) whose inner and outer radii are R_o and R_n, and which is subjected to internal and external pressures P_o and P_n respectively. Also let t, p and q be the circumferential, radial and longitudinal stresses, and e_t, e_p and e_q the circumferential, radial

and longitudinal strains, at the point O, E being the modulus of elasticity of the material. And let T_o and T_n be the circumferential tensions at the inner and outer surfaces, or the values of t when $r = R_o$ and when $r = R_n$.

In the strained cylinder, the principal stresses at the point O are evidently the radial pressure p, which varies in value from P_o at the inner to P_n at the outer surface; the circumferential tension t, which varies from T_o at the inner to T_n at the outer surface; and the longitudinal stress q, which is zero in the particular case considered but which might be either tension or compression and either constant or variable. From equations (2), therefore, we obtain as the values of the principal strains,

$$\left.\begin{aligned} e_t &= \frac{1}{E}\left(t + \frac{p}{3}\right) \\ e_p &= -\frac{1}{E}\left(p + \frac{t}{3}\right) \\ e_q &= -\frac{1}{E}\left(\frac{t}{3} - \frac{p}{3}\right) \end{aligned}\right\} \quad (3)$$

and since, under the stated conditions, e_q is constant,

$$* \; t - p = \text{constant} = k \qquad (4)$$

Furthermore, the total force tending to split the cylinder lengthwise (dividing it into two parts by a diametral plane) is $2(P_o R_o - P_n R_n)$; and the resistance to such a rupture is the total stress on a section of the cylinder made by a diametral plane, which total stress is $2\int_{R_o}^{R_n} t\,dr$.† Thus we have

$$\int_{R_o}^{R_n} t\,dr = P_o R_o - P_n R_n \qquad (5)$$

and, assuming $t = f'(r)$, this gives $f(r)\Big]_{R_o}^{R_n} = P_o R_o - P_n R_n$,

from which we see that $f(r) = -pr \pm$ constant, so that the value of $t = f'(r)$ is given by

$$t = -p - r\frac{dp}{dr} \qquad (6)$$

* It should be noted that this same result, $t - p =$ constant, follows when q is constant as well as when q is zero.

† We here assume the cylinder to be of unit length.

Combining (6) with (4) we obtain

$$2p + k = -r\frac{dp}{dr} \tag{7}$$

and the integration of this gives

$$2p + k = \frac{k_1^2}{r^2} \tag{8}$$

in which k_1 is a constant of integration.

Finally, eliminating k from (8) by means of (4),

$$t + p = \frac{k_1^2}{r^2} \tag{9}$$

17. Equations (4) and (9) express what are known as *Lamé's Laws:* *

1. At any point whatever in a cylinder under fluid pressure the sum of the circumferential tension and the radial pressure varies inversely as the square of the radius.

2. The difference of the circumferential tension and the radial pressure is the same at all points.

These, then, are the equations which express the relation between the circumferential tension and the radial pressure at all points within the cylinder walls:

$$\left.\begin{aligned} t - p = k = T_o - P_o = T_n - P_n \\ (t + p)r^2 = k_1^2 = (T_o + P_o)\,R_o^2 = (T_n + P_n)\,R_n^2 \end{aligned}\right\} \tag{10}$$

18. Eliminating T_n between the last parts of equations (10), we have

$$T_o = P_o\,\frac{R_n^2 + R_o^2}{R_n^2 - R_o^2} - P_n\,\frac{2\,R_n^2}{R_n^2 - R_o^2}$$

and substituting this value of T_o in the first parts of the same equations, we have, after combining:

$$t = \frac{P_o R_o^2 - P_n R_n^2}{R_n^2 - R_o^2} + \frac{R_o^2 R_n^2 (P_o - P_n)}{R_n^2 - R_o^2}\,\frac{1}{r^2} \tag{11}$$

$$p = -\frac{P_o R_o^2 - P_n R_n^2}{R_n^2 - R_o^2} + \frac{R_o R_n^2 (P_o - P_n)}{R_n^2 - R_o^2}\,\frac{1}{r^2} \tag{12}$$

and these equations enable us to determine the values of t and of p at any point.

* As explained, these laws are only strictly true when the longitudinal stress is constant, or zero.

19. To determine the principal strains at any point, we have only to substitute in (3) the values of the principal stresses (t and p) as given in (11) and (12), thus obtaining

$$e_t = \frac{1}{E}\left[\frac{2}{3}\,\frac{P_o R_o^2 - P_n R_n^2}{R_n^2 - R_o^2} + \frac{4}{3}\,\frac{R_o^2 R_n^2 (P_o - P_n)}{R_n^2 - R_o^2}\,\frac{1}{r^2}\right] \quad (13)$$

$$e_p = \frac{1}{E}\left[\frac{2}{3}\,\frac{P_o R_o^2 - P_n R_n^2}{R_n^2 - R_o^2} - \frac{4}{3}\,\frac{R_o^2 R_n^2 (P_o - P_n)}{R_n^2 - R_o^2}\,\frac{1}{r^2}\right] \quad (14)$$

$$e_q = -\frac{1}{E}\left[\frac{2}{3}\,\frac{P_o R_o^2 - P_n R_n^2}{R_n^2 - R_o^2}\right] \quad (15)$$

The first two of these equations are the fundamental ones from which we shall deduce all the formulæ used in our study of the elastic strength of guns.

The greatest of the three strains given by (13), (14) and (15) for any point in the cylinder walls must not at any time exceed the elastic limit of strain of the material of the cylinder. That is, calling θ the elastic limit of the material, as determined in a testing machine, the limiting value for each of the three strains e_t, e_p and e_q is $\frac{\theta}{E}$.

As e_t and e_p denote the general values of the circumferential and radial strains (at any radius r), we shall distinguish the values of the circumferential and radial strains at radius R_o by $e_t(R_o)$ and $e_p(R_o)$, and those at radius R_n by $e_t(R_n)$ and $e_p(R_n)$.

20. The quantities Ee_t, Ee_p and Ee_q, respectively, equal in value the simple stresses which, acting alone, would cause the strains e_t, e_p and e_q, but these strains are actually caused by the concurrent action of the two stresses p and t. We shall hereafter designate Ee_t, Ee_p and Ee_q as the *true stresses*, circumferential, radial and longitudinal respectively.

21. The distribution of the true stresses throughout the walls of a simple cylinder under fluid pressure is best shown graphically, and we will therefore do this for three cases; first, when the outer pressure (P_n) is zero; second, when the inner pressure (P_o) is zero; and third, when both pressures act and P_o is greater than P_n. In each case we assume a cylinder whose outer is three times its inner radius ($R_n = 3\,R_o$), so that its walls are a caliber thick.

22. Case I.—No Exterior Pressure.—Putting $P_n = 0$ and $R_n = 3R_o$ in (13), (14) and (15), we obtain as the values of the true stresses:

$$\left.\begin{aligned} Ee_t &= \frac{P_o}{12}\left(1 + \frac{18R_o^2}{r^2}\right) \\ Ee_p &= \frac{P_o}{12}\left(1 - \frac{18R_o^2}{r^2}\right) \\ Ee_q &= -\frac{P_o}{12} \end{aligned}\right\} \quad (16)$$

From these it will be seen that as r increases from R_o to R_n the circumferential true stress diminishes from $\frac{19}{12}P_o$ to $\frac{3}{12}P_o$, its value midway, where $r = 2R_o$, being $\frac{11}{24}P_o$; the radial true stress

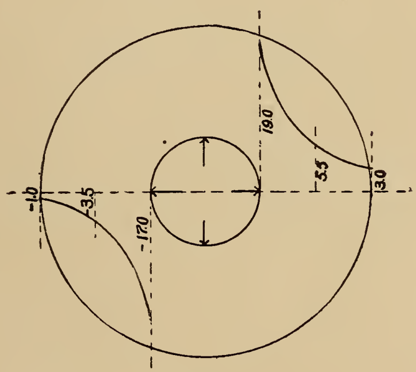


Fig. 3.

diminishes (algebraically it increases) from $-\frac{17}{12}P_o$ to $-\frac{1}{12}P_o$, its value midway being $-\frac{7}{24}P_o$; while the longitudinal true stress has the constant value $-\frac{1}{12}P_o$ throughout the cylinder wall. Figure 3 illustrates the distribution of the tangential and radial true stresses, the former on the right and the latter on the left of the section, the ordinates above the horizontal diameter indicating tensions and those below it indicating compressions. The figures on the inner, middle and outer ordinates are the true stresses in tons per square

inch which would result from an internal pressure of 12 tons per square inch.

23. Case II.—No Interior Pressure.—Putting $P_o = 0$ and $R_n = 3R_o$ in (13), (14) and (15) we obtain as the values of the true stresses,

$$\left.\begin{aligned} Ee_t &= -\frac{3P_n}{4}\left(1+\frac{2R_o^2}{r^2}\right) \\ Ee_p &= -\frac{3P_n}{4}\left(1-\frac{2R_o^2}{r^2}\right) \\ Ee_q &= +\frac{3P_n}{4} \end{aligned}\right\} \quad (17)$$

From these it will be seen that as r increases from R_o to R_n, the circumferential true stress diminishes (algebraically it increases)

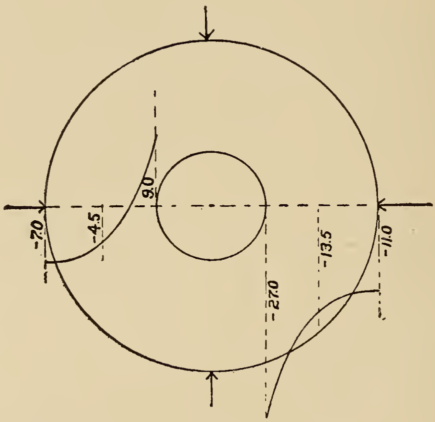


FIG. 4.

from $-\frac{9}{4}P_n$ to $-\frac{11}{12}P_n$, its value midway, where $r = 2R_o$, being $-\frac{9}{8}P_n$; the radial true stress diminishes from $+\frac{3}{4}P_n$ to $-\frac{7}{12}P_n$, its midway value being $-\frac{3}{8}P_n$; while the longitudinal true stress has the constant value $+\frac{3}{4}P_n$. Figure 4 illustrates this, the right-hand curve showing the tangential and the left-hand curve the radial true stress at each point in the wall thickness, ordinates

above the horizontal diameter indicating tensions and those below it indicating compressions. The figures on the inner, middle and outer ordinates are the true stresses in tons per square inch which would result from an external pressure of 12 tons per square inch.

24. Case III.—Exterior Pressure One-half the Interior Pressure. —Putting $P_n = \frac{1}{2} P_o$ and $R_n = 3R_o$ in (13), (14) and (15), we obtain as the values of the true stresses:

$$\left.\begin{aligned} Ee_t &= +\frac{P_o}{24}\left(18\,\frac{R_o^2}{r^2} - 7\right) \\ Ee_p &= -\,\frac{P_o}{24}\left(18\,\frac{R_o^2}{r^2} + 7\right) \\ Ee_q &= +\frac{7}{24}\,P_o \end{aligned}\right\} \quad (18)$$

From these it will be seen that as r increases from R_o to R_n the circumferential true stress diminishes from $\frac{11}{24} P_o$ to $-\frac{5}{24} P_o$, its value midway being $-\frac{5}{48} P_o$; the radial true stress diminishes (algebraically it increases) from $-\frac{25}{24} P_o$ to $-\frac{9}{24} P_o$, its value midway being $-\frac{23}{48} P_o$; while the longitudinal true stress has the constant value $+\frac{7}{24} P_o$. Figure 5 illustrates the distribution of

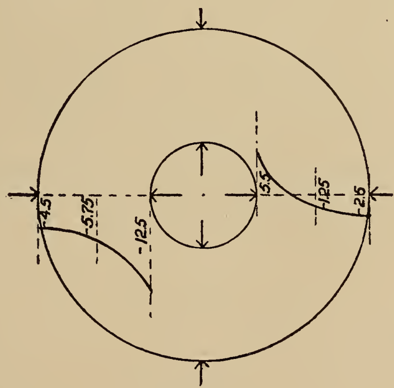


FIG. 5.

the tangential and radial true stresses, the former on the right and the latter on the left of the section, the ordinates above the longitudinal diameter indicating tensions and those below it indicating

compressions. The figures on the inner, middle and outer ordinates are the true stresses in tons per square inch which would result from an internal pressure of 12 tons per square inch and an external pressure of 6 tons per square inch.

25. Comparing Figure 5 with Figures 3 and 4, it will be seen that the ordinates of the curves in the former are the algebraic sums of the corresponding ordinates of Figure 3 and half those of Figure 4; the stresses due to 12 tons internal and 6 tons external pressure acting together are the same as the algebraic sums of the stresses due to the same pressures acting separately.

26. Since the strains given by equations (13), (14) and (15) are changes of length per unit length, the change of thickness of the cylinder wall may be determined in any case by integrating $e_p \, dr$ between the limits R_n and R_o. But the change of radius at any point whose radius is r must be re_t, and the difference between the change of the outer radius $(R_n \; e_t(R_n))$ and the change of the inner radius $(R_o \; e_t(R_o))$ must equal the change of thickness. Therefore $\int_{R_o}^{R_n} e_p \, dr = R_n \; e_t(R_n) - R_o \; e_t(R_o)$, and it will be found upon trial that this is true of the values of c_p and e_t given by (13) and (14).

27. The hypothesis made in **15** that there is no longitudinal stress, is, of course, not true, as a rule, for actual constructions. In the built-up guns, for example, whose strength we are investigating, one end of the bore is closed by a breech-block which sustains the internal pressure and thus causes a total longitudinal stress $\pi R_o^2 P_o$ which is distributed over the cross-section of one or more of the cylinders of which the gun is composed. This stress may be taken account of by assuming that it is uniformly distributed, but, as will be shown further on, the hypothesis that q is zero accords as well or better with the facts than any other available one.

EXAMPLES II.

(1) Show that in an infinitely thick hollow cylinder $(R_n = \infty)$ subjected only to internal pressure (P_o) the true circumferential and radial stresses at the inner surface are of equal value but opposite sign. What are their values? What is the value of the longitudinal stress?

$$+\frac{4}{3} P_o; \; -\frac{4}{3} P_o; \; 0.$$

(2) What are the true stresses at the inner surface of an infinitely thick hollow cylinder subjected only to external pressure (P_n)?

$$-2P_n;\ +\frac{2}{3}P_n;\ +\frac{2}{3}P_n.$$

(3) If the external and internal pressures are equal, what is the state of stress in the cylinder walls?

$$Ee_t = Ee_p = -\frac{2}{3}P_o;\ Ee_q = +\frac{2}{3}P_o.$$

(4) What would be the change of thickness of a hollow cylinder one diameter thick under internal pressure alone?

$$-\frac{5}{6}\frac{P_o R_o}{E}.$$

(5) What would be the change of thickness of a hollow cylinder one diameter thick under external pressure only?

$$-\frac{P_n R_o}{2E}.$$

(6) A hollow cylinder half a caliber thick is subjected to an internal pressure of 6 tons per square inch. What is the greatest true stress resulting and where does it occur? What are the true stresses at the outer surface?

$$Ee_t(R_o) = 12 \text{ tons per sq. in.}$$
$$Ee_t(R_n) = 4;\ Ee_p(R_n) = -1\tfrac{1}{3};\ Ee_q = -1\tfrac{1}{3}.$$

(7) If the cylinder of Example (6) is only one quarter of a caliber thick, what are the true stresses at inner and outer surfaces?

$$\left.\begin{array}{l} Ee_t(R_o) = 17\tfrac{3}{5};\ Ee_p(R_o) = -11\tfrac{1}{5} \\ Ee_t(R_n) = 9\tfrac{3}{5};\ Ee_p(R_n) = -\ 3\tfrac{1}{5} \end{array}\right\}\ Ee_q = -3\tfrac{1}{5}.$$

(8) Show that, as the thickness of wall of a cylinder under internal pressure is made a smaller and smaller fraction of its inner diameter, the circumferential stress becomes more and more nearly constant throughout the wall. If the circumferential stress were constant, what would be the relation between it and the internal pressure?

$$P_o = \frac{R_n - R_o}{R_o}T.$$

(9) A hollow steel tube, radii 3 in. and 6 in., is subjected to an internal pressure of 13 tons per sq. in. Determine the three principal strains at the inner surface. What is the least elastic limit

of the steel which will permit the application of such a pressure without permanent set?

$$e_t(R_o) = .002;\ e_p(R_o) = -.00156;\ e_q = -.00022.$$

26 tons per sq. in.

(10) With the data of Example (9) determine the three principal strains at the outer surface of the tube.

$$e_t(R_n) = .00067;\ e_p(R_n) = -.00022;\ e_q = -.00022.$$

(11) Show that the change of wall thickness of a cylinder is independent of the value of the external pressure in the case where the outer radius is twice the inner radius.

$$\text{Change} = -\frac{2P_o R_o}{3E}.$$

CHAPTER III.

THE ELASTIC STRENGTH OF SIMPLE HOLLOW CYLINDERS.

28. We will denote the elastic limit under tension of the material of the cylinder by θ and its elastic limit under compression by ρ. In the case of the forged steel used in modern gun construction, these elastic limits are usually taken to be equal, but with some materials, notably cast iron, ρ is considerably greater than θ, and even in the case of steel it is probable that ρ is always somewhat greater than θ.

In accordance with the principle stated in **12**, we consider that the limit of safety is reached whenever either of the principal strains, circumferential, radial or longitudinal, attains the value $\frac{\theta}{E}$ in extension or the value $\frac{\rho}{E}$ in compression; in either case we suppose that the strain ceases to be wholly elastic, and though rupture may not follow, some permanent change of dimensions or distortion will result.

In order, therefore, to determine the maximum pressure which a given cylinder will withstand without permanent set, we have only to equate the greatest strain of extension which results from the pressure to $\frac{\theta}{E}$ and the greatest strain of compression to $\frac{\rho}{E}$ and the least of the pressures given by solving these two equations is the greatest pressure which the cylinder can safely be subjected to. In other words, the limit of the elastic strength of the cylinder is reached when either the greatest true stress of tension equals the elastic limit of the material under simple tension, or the greatest true stress of compression equals the elastic limit of the material under simple compression.

29. Internal Pressure Only.—Putting $P_n = 0$ in **(13)** we obtain

$$Ee_t = \frac{2P_oR_o^2}{3(R_n^2 - R_o^2)}\left(1 + \frac{2R_n^2}{r^2}\right) \tag{19}$$

This is always plus, showing that the circumferential true stress is always tension; and its greatest value is when r has its least

value R_o. Hence we find the value of P_o which will make the greatest circumferential true stress equal the elastic limit of the material by putting $Ee_t = \theta$ and $r = R_o$ in (19), This gives

$$\theta = \frac{2P_o}{3(R_n^2 - R_o^2)}(R_o^2 + 2R_n^2)$$

$$P_o = \frac{3(R_n^2 - R_o^2)}{4R_n^2 + 2R_o^2}\theta \qquad (20)$$

Next putting $P_n = 0$ in (14), we obtain

$$Er_p = \frac{2P_o R_o^2}{3(R_n^2 - R_o^2)}\left(1 - \frac{2R_n^2}{r^2}\right) \qquad (21)$$

This is always negative, showing that the radial true stress is always compression; and its greatest value (numerically) is when $r = R_o$. Hence we find the value of P_o which will make the greatest radial true stress equal the elastic limit of the material by putting $Ee_p = -\rho$ and $r = R_o$ in (21). This gives

$$-\rho = \frac{2P_o}{3(R_n^2 - R_o^2)}(R_o^2 - 2R_n^2)$$

$$P_o = \frac{3(R_n^2 - R_o^2)}{4R_n^2 - 2R_o^2}\rho \qquad (22)$$

The determination of the value of the longitudinal true stress is unnecessary, since it can never exceed, and in all practical cases is much less than, one or the other of the other two principal true stresses, the circumferential and the radial.

Now, comparing (20) and (22), since ρ is always equal to or greater than θ, and since the denominator of (20) is greater than the denominator of (22), the value of P_o given by (20) will always be less than the value of P_o given by (22). When P_o reaches the value given by (20), the elastic limit of strain is reached circumferentially, and further increase of P_o is inadmissible.

Consequently the maximum internal pressure allowable in the case of a simple hollow cylinder under no exterior pressure is given by

$$P_o = \frac{3(R_n^2 - R_o^2)}{4R_n^2 + 2R_o^2}\theta \qquad (20 \text{ bis})$$

in which θ is the elastic limit of the material under tension.

Evidently equation (20) gives not only the relation between the maximum allowable internal pressure and the elastic limit of the

material, but equally the relation between any internal pressure and the greatest resulting true stress (within the elastic limit). Moreover, by means of (20) the necessary thickness of a cylinder to safely withstand a given internal pressure is readily determined, since, solving for R_n, we have $R_n = R_o \sqrt{\frac{3\theta + 2P_o}{3\theta - 4P_o}}$.

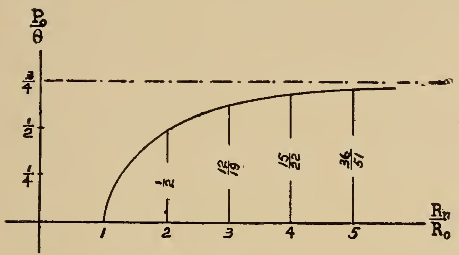


FIG. 6.

Figure 6 shows how the ratio $\frac{P_o}{\theta}$ increases with the ratio $\frac{R_n}{R_o}$, attaining the maximum value $\frac{3}{4}$ when $\frac{R_n}{R_o} = \infty$, and clearly indicates the small effect upon strength of increasing wall thickness beyond a caliber.

30. Examples.—(1) What is the limiting value of the internal pressure which any simple cylinder (regardless of its thickness) will stand without permanent set, the elastic limit of its material being θ? $\frac{3}{4}\theta$.

(2) The walls of a 6-inch steel shell are 1.5 in. thick; if the tensile strength of the steel is 50 tons per sq. in., what powder pressure will burst the shell? 25 tons per sq. in.

(3) What internal pressure will produce a circumferential elongation of .0015 in the case of a simple steel tube of 3 in. interior and 6 in. exterior radius? 9.75 tons per sq. in.

(4) What internal pressure will a cast-steel cylinder of 4 in. internal and 6 in. external radius stand within its elastic limit of 30,000 lbs. per sq. in.? 10,227 lbs. per sq. in.

(5) A nickel-steel cylinder of 7 in. interior radius and 0.5 in. wall thickness has an elastic limit of 70,000 lbs. per sq. in. What internal pressure will it withstand? 4713 lbs. per sq. in.

(6) A cylinder of 7 in. interior diameter has walls 3.5 inches thick. If its elastic limit is 36,000 lbs. per sq. in., what internal pressure will it stand? How much pressure could it withstand if its wall thickness were doubled? if trebled?

18,000; 22,740; 24,550 lbs. per sq. in.

(7) Determine the proper thickness for a cylinder of 6 in. inner radius which is to stand an internal pressure of 3000 lbs. per sq. in., the elastic limit of the material being 28,000 lbs. per sq. in. 0.708 in.

(8) If the radii are 8 in. and 9 in. and the elastic limit is 60,000 lbs. per sq. in., what is the maximum allowable internal pressure? What would it be if the circumferential stress were constant throughout the cylinder walls?

6770; 7500 lbs. per sq. in.

(9) What thickness should a cylinder of 4 in. interior radius have to withstand an internal pressure of 8000 lbs. per sq. in., if the elastic limit is 40,000 lbs. per sq. in.? 0.973 in.

(10) What internal pressure will a cylinder of 6 in. interior radius and 4 in. wall thickness withstand, if the elastic limit is 18 tons per sq. in.? 7.32 tons per sq. in.

31. External Pressure Only.—Putting $P_o = 0$ in (13) we obtain

$$Ee_t = -\frac{2P_n R_n^2}{3(R_n^2 - R_o^2)}\left(1 + \frac{2R_o^2}{r^2}\right) \tag{23}$$

This is always negative, showing that the circumferential true stress is always compression; and its greatest value is when $r = R_o$. Hence we find the value of P_n which will make the greatest circumferential true stress equal the elastic limit of the material by putting $Ee_t = -\rho$ and $r = R_o$ in (23). This gives

$$\rho = \frac{2P_n R_n^2}{R_n^2 - R_o^2}$$

$$P_n = \frac{R_n^2 - R_o^2}{2R_n^2}\rho \tag{24}$$

Next, putting $P_o = 0$ in (14), we obtain

$$Ee_p = -\frac{2P_n R_n^2}{3(R_n^2 - R_o^2)}\left(1 - \frac{2R_o^2}{r^2}\right) \tag{25}$$

This is positive when $r = R_o$ and continues so until r attains the value $R_o\sqrt{2}$, beyond which point it becomes negative; its greatest numerical value, however, is when $r = R_o$. Hence, to find the value of P_n which would make the greatest radial true stress equal the elastic limit of the material, we would put $Ee_p = \theta$ and $r = R_o$ in (25). A comparison of (25) with (23), however, will show that, for every value of r, Ee_t is greater than Ee_p, so that the elastic strength of the cylinder depends upon its resistance to circumferential stress and not upon its resistance to radial stress.*

Consequently the maximum external pressure allowable in the case of a simple hollow cylinder under no interior pressure is given by

$$P_n = \frac{R_n^2 - R_o^2}{2R_n^2}\rho \qquad (24\ \text{bis})$$

in which ρ is the elastic limit of the material under compression.

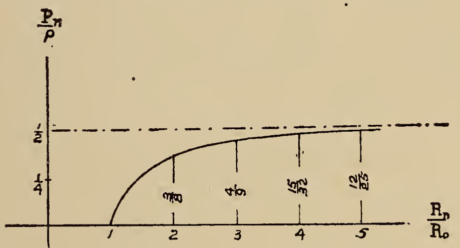


FIG. 7.

Figure 7 shows the increase of the ratio $\frac{P_n}{\rho}$ as wall thickness increases, and clearly indicates how little is gained by going beyond a thickness of one caliber.

Of course (24) expresses the relation between the external pressure and the greatest resulting true stress within as well as at the limit of elastic strain.

32. Examples.—(1) What is the limiting value of the external pressure which any simple hollow cylinder, regardless of its thickness, can withstand without permanent set, the elastic limit of compression being ρ? $\frac{1}{2}\rho$.

* With a material like cast iron, of which the elastic limit of compression greatly exceeds that of tension, the limit of elastic strain radially (in this case extension) may in some cases be reached before the limit of elastic strain circumferentially (in this case compression) is attained.

(2) What external pressure can a tube of 7.5 in. interior radius and 1.75 in. thickness of wall withstand, the elastic limit for compression being 30,000 lbs. per sq. in.? 5139 lbs. per sq. in.

(3) How thick should the tube of Example (2) ($R_o = 7.5$ in.) be to withstand an external pressure of 10,000 lbs. per sq. in.? 5.49 in.

(4) The inner and outer radii of a steel tube are 4 in. and 7 in., and it is to be subjected to an external pressure of 8.3 tons per sq. in. What are the circumferential and radial strains at the inner surface? What is the greatest true stress?

$-.001897$; $+.000632$; 24.65 tons per sq. in.

(5) How thick should the walls of a 6-inch shell be to withstand 6 tons per sq. in. external pressure, without passing the elastic limit of compression of 18 tons per sq. in.? 1.27 in.

(6) What external pressure will a cylinder of 6 in. interior radius and 4 in. wall thickness withstand, if the elastic limit is 18 tons per sq. in.? 5.76 tons per sq. in.

(7) What wall thickness should a cylinder have to withstand 8000 lbs. per sq. in. external pressure, the interior radius being 4 in. and the elastic limit 40,000 lbs. per sq. in.? 1.16 in.

(8) If the interior radius is 8 in., the wall thickness 1 in., and the elastic limit 60,000 lbs. per sq. in., what is the maximum allowable external pressure? What would it be if the circumferential stress were constant throughout the walls?

6297; 6667 lbs. per sq. in.

33. Both Internal and External Pressure.—Which of the true stresses first reaches the elastic limit depends, in this case, upon the relation between the two pressures, and we must consider the three possible cases separately.

$$P_n > P_o$$

In this case both terms of the value of Ee_t, equation (13), are negative, showing that the circumferential true stress is always compression; and its greatest numerical value is when r has its least value R_o. On the other hand, the two terms of the value of Ee_p, equation (14), have opposite signs, showing that the radial true stress may be either tension or compression, according to which term preponderates, and also showing that at each point Ee_t is

greater numerically than Ee_p. We therefore obtain an equation between the values of P_o and P_n which will make the greatest true stress resulting from their concurrent action equal the elastic limit of the material by putting $Ee_t = -\rho$ and $r = R_o$ in (13). This gives

$$\rho = \frac{6P_n R_n^2 - 2P_o R_o^2 - 4P_o R_n^2}{3(R_n^2 - R_o^2)}$$

$$P_o = \frac{6P_n R_n^2 - 3(R_n^2 - R_o^2)\rho}{4R_n^2 + 2R_o^2} \tag{26}$$

Consequently, *when* P_n *exceeds* P_o, the relation between the internal pressure and the maximum allowable external pressure is given by (26), in which ρ is the elastic limit under compression.

$$P_o > P_n \text{ but } P_o R_o^2 < P_n R_n^2.$$

In this case the first term of the value of Ee_t, equation (13), remains negative, while the second term is positive, so that the circumferential true stress may be either tension or compression, according to which term preponderates. But both terms of the value of Ee_p, equation (14), are now negative, showing that the radial true stress is always compression and is numerically greater than Ee_t at each point; moreover, the maximum numerical value of Ee_p is when r has its least value R_o. We therefore obtain an equation between the values of P_o and P_n which will make the greatest true stress resulting from their concurrent action equal to the elastic limit of the material by putting $Ee_p = -\rho$ and $r = R_o$ in (14). This gives

$$\rho = \frac{P_o(4R_n^2 - 2R_o^2) - 2P_n R_n^2}{3(R_n^2 - R_o^2)}$$

$$P_o = \frac{3(R_n^2 - R_o^2)\rho + 2P_n R_n^2}{4R_n^2 - 2R_o^2} \tag{27}$$

Consequently, *when* P_o *exceeds* P_n *but at the same time* $P_oR_o^2$ *is less than* $P_nR_n^2$, the maximum allowable internal pressure is given by (27), in which ρ is the elastic limit of the material under compression.

$$P_o > P_n \text{ and } P_o R_o^2 > P_n R_n^2$$

In this case both terms of the value of Ee_t, equation (13), are positive, showing that the circumferential true stress is always tension; its greatest value occurs when $r = R_o$; and at each point it is

numerically greater than Ee_p since the two terms which make up the latter's value are now of different signs. We therefore obtain an equation between the values of P_o and P_n which will make the greatest true stress resulting from their concurrent action equal the elastic limit of the material by putting $Ee_t = \theta$ and $r = R_o$ in (13). This gives

$$\theta = \frac{P_o(4R_n^2 + 2R_o^2) - 6P_n R_n^2}{3(R_n^2 - R_o^2)}$$

$$P_o = \frac{3(R_n^2 - R_o^2)\theta + 6P_n R_n^2}{4R_n^2 + 2R_o^2} \qquad (28)$$

Consequently, *when* P_o *exceeds* P_n *and at the same time* $P_o R_o^2$ *exceeds* $P_n R_n^2$, the maximum allowable internal pressure is given by (28), in which θ is the elastic limit of the material under tension.

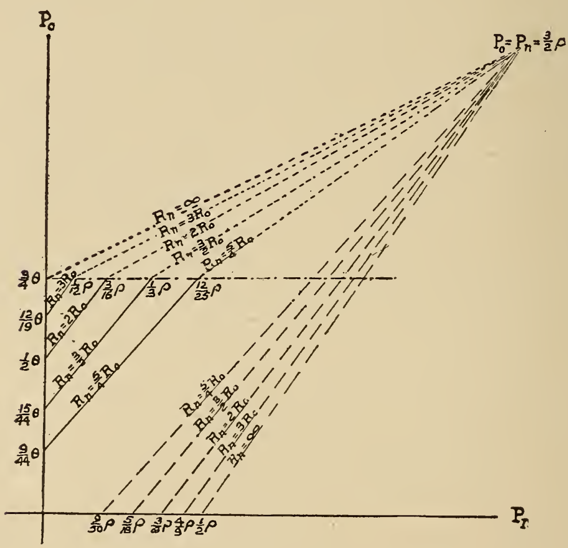


FIG. 8.

Of course equations (26), (27) and (28), each under its appropriate conditions, express the relation between the internal pressure, the external pressure and the greatest resulting true stress within the elastic limit as well as at that limit.

34. The relation between simultaneous values of P_o and P_n which will just bring a given cylinder to the limit of its elastic strength

may be graphically shown by drawing the three straight lines represented by equations (26), (27) and (28). In Figure 8, values of P_o are represented by the ordinates, and corresponding values of P_n by the abscissæ, and the cases of five different thicknesses of cylinder wall are illustrated.

Taking (28) first, when $P_n = 0$, the value of P_o is $\frac{3(R_n^2 - R_o^2)}{4R_n^2 + 2R_o^2}\,\theta$, approaching $\frac{3}{4}\theta$ as a limit when the thickness of the cylinder is indefinitely increased. If P_n is given successively increasing values, the maximum allowable value of P_o also increases, the relation between them being given by (28) and represented by the full lines of the diagram. When P_n attains the value $\frac{3R_o^2}{4R_n^2}\,\theta$, P_o has the value $\frac{3}{4}\theta$, regardless of the thickness of the cylinder, and with these values of the pressures the inner surface of the cylinder is both at its elastic limit of extension circumferentially and at its elastic limit of compression radially.

Further increase of P_o is allowable if P_n be also increased, but from the point where $P_o = \frac{3}{4}\theta$ the relation between P_o and P_n is given by (27), and represented by the dotted lines of the diagram. When P_o reaches the value $\frac{3}{2}\,\rho$, P_n must also equal $\frac{3}{2}\,\rho$, regardless of the thickness of the cylinder wall, and no value of P_n will enable P_o to exceed $\frac{3}{2}\,\rho$, nor will any value of P_o enable P_n to exceed $\frac{3}{2}\,\rho$, without the elastic strength of the cylinder being exceeded. At this point, where $P_o = P_n = \frac{3}{2}\,\rho$, the inner surface of the cylinder is at its elastic limit of compression both radially and circumferentially.

If now P_o be gradually reduced, while P_n is kept as great as allowable, the relation between the values of P_o and P_n will be given by (26) and is represented by the dash lines of the diagram. When P_o has been reduced to zero, the inner surface of the cylinder being maintained at the elastic limit of compression circumferentially, P_n has the value $\frac{R_n^2 - R_o^2}{2R_n^2}\,\rho$, approaching $\frac{\rho}{2}$ as a limit when the thickness of the cylinder wall is indefinitely increased.

It will be observed that the full and dotted lines represent the

relation between P_o and P_n when the former is as great as allowable; while the dash lines represent the relation between P_o and P_n when the latter is as great as allowable.

35. Examples.—(1) If $R_n = 2R_o$ and $P_n = \rho = \theta$, what is the greatest and what the least allowable value of P_o? $\frac{17}{14}\rho$; $\frac{5}{6}\rho$.

(2) If $R_n = \frac{5}{4}R_o$ and $P_n = \rho$, what are the greatest and least allowable values of P_o? $\frac{77}{68}\rho$; $\frac{41}{44}\rho$.

(3) If $R_n = \frac{3}{2}R_o$, what value must P_n have in order that P_o may have the value of $\frac{3}{4}\theta$? $\frac{1}{3}\theta$ or $\frac{8}{9}\theta$.

(4) If $R_n = \frac{5}{4}R_o$, what value must P_n have in order that P_o may have the value $\frac{3}{4}\theta$? $\frac{12}{25}\theta$ or $\frac{21}{25}\theta$.

(5) What internal pressure will a cast-steel cylinder of 4 in. interior and 6 in. exterior radius stand within its elastic limit of 30,000 lbs. per sq. in. if it is under an external pressure of 5000 lbs. per sq. in.? 16,363 lbs. per sq. in.

(6) A nickel-steel cylinder of 7 in. interior radius and 1.5 in. wall thickness has an elastic limit of 70,000 lbs. per sq. in. What external pressure will it withstand if it is under an internal pressure of 10,000 lbs. per sq. in.? 20,190 lbs. per sq. in.

(7) The inner and outer radii of a steel tube are 4 in. and 7 in.: what external pressure will enable it to withstand an internal pressure of 20,000 lbs. per sq. in., if the elastic limit of the steel is 36,000 lbs. per sq. in.? 3390 to 27,630 lbs. per sq. in.

CHAPTER IV.

THE ELASTIC STRENGTH OF COMPOUND CYLINDERS.

36. A reference to Figure 3 will show that the outer portions of a thick simple cylinder play but a small part in resisting internal pressure. A *compound cylinder* is one formed by the superposition of simple cylinders, the object being to utilize to the utmost the contractile power of the outer parts and thus to increase the resistance to internal pressure beyond what it would be if the entire mass were in one piece.

If the elementary cylinders are of the same material, or have equal moduli of elasticity, they must be assembled so that each exerts an initial pressure upon the one within it. This is accomplished by making the interior diameter of each elementary cylinder (before it is put in place) less than the exterior diameter of the cylinder upon which it is to be superposed by a certain quantity which is called the *shrinkage*. A compound cylinder so assembled is said to be under *initial tension*.

If the elementary cylinders are of different materials, and are so arranged that the modulus of elasticity of each is greater than that of the one within it, they may be assembled without shrinkage. Such a cylinder is called a compound cylinder of *variable elasticity*.

These two principles of variable elasticity and of initial tension were formerly often employed in combination, the commonest examples being cast-iron guns with reinforcing hoops of steel, but in modern gun construction, excepting for certain bronze field pieces, steel is now used to the exclusion of other metals, and the principle of initial tension is universally adopted.*

37. In the investigation of the elastic strength of a compound

* Rodman was to some degree successful in applying the principle of initial tension to solid guns, the cast iron smooth bore guns known by his name having been cast hollow and cooled from the interior with the object of securing compression of the bore and tension of the outer parts of the finished gun; and the application of essentially the same process to steel guns, either cast or forged in one piece, has been shown to be feasible and advantageous.

cylinder, it is necessary to consider its state of strain both when the maximum internal pressure is acting and when the internal pressure is zero: the first of these two conditions is called the *state of action* and the second is called the *state of rest.*

In the state of action each cylinder except the outer one is subjected to two pressures, one internal and the other external, while the outer cylinder is subjected to internal pressure only, atmospheric pressure being neglected on account of its insignificant value as compared with the other forces.

In the state of rest the inner cylinder is under external pressure only, the outer cylinder is under internal pressure only, and each of the intermediate cylinders is subjected to both an internal and an external pressure.

38. We adopt the following nomenclature:

R_o and R_1 are the inner and outer radii of the innermost or 1st elementary cylinder, R_1 and R_2 of the next, ... , R_{n-1} and R_n of the outermost or nth.

θ_o and ρ_o, θ_1 and ρ_1, ... θ_n and ρ_n are the elastic limits of the material of the elementary cylinders in the same order, from the 1st to the nth; and E is their common modulus of elasticity.

P_o, P_1, ... P_n are the radial stresses in the state of action at the successive surfaces of the elementary cylinders, and $\overline{P}_o$, $\overline{P}_1$, ... $\overline{P}_n$ are the radial stresses at the same surfaces in the state of rest; they are always plus, excepting that $\overline{P}_o$, P_n and $\overline{P}_n$, being only atmospheric pressures, are considered to be zero.*

T_o, T_1, ... T_n are the circumferential stresses in the state of action, and $\overline{T}_o$, $\overline{T}_1$, ... $\overline{T}_n$ are the circumferential stresses in the state of rest, at the successive surfaces whose radii are R_o, R_1, ... R_n; they are plus when tensions and minus when compressions.

$e_p(R_o)$, $e_p(R_1)$, ... $e_p(R_n)$ are the radial strains, and $e_t(R_o)$, $e_t(R_1)$, ... $e_t(R_n)$ are the circumferential strains at radii R_o, R_1, ... R_n, in the state of action; the same symbols with a dash over each, as $\overline{e}_p(R_o)$, are the corresponding strains in the state of rest; they are all plus when lengthenings and minus when shortenings.

* This convention that radial stresses which are compressive shall be called positive, is explained in 14: it must be remembered, however, that a radial strain, like all other strains, is called minus when it denotes a decrease of length.

Since the states of stress and strain on either side of the surface of contact of two elementary cylinders may be different (must be if they were assembled with shrinkage), it is necessary to distinguish between them. A prime mark over any letter or symbol indicates that it refers to the outer of the two surfaces which are united by the contact. Thus T_1' is the tension at the inner surface of the second cylinder as distinguished from T_1 which is the tension at the outer surface of the first cylinder; $e_p(R_2')$ is the radial strain in the outer of the two surfaces which meet at R_2; $Ee_t(R_1')$ and $Ee_t(R_1)$ are the circumferential true stresses in the outer and inner of the two surfaces which meet at R_1; and so on. (At R_o and R_n no prime marks are needed, as there is but one surface at each.)

$p_o, p_1, \ldots\ldots p_n$ are the simultaneous changes in the radial pressures $P_o, P_1, \ldots\ldots P_n$ resulting from any cause, such, for example, as the cessation of the internal pressure P_o.

39. Evidently, with any given assemblage of elementary cylinders, the elastic strength to resist internal pressure will be greatest when in the state of action each cylinder is strained to its elastic limit. Moreover, in a compound cylinder so assembled that all the elementary cylinders reach their elastic limits of strain simultaneously under the action of the internal pressure P_o, that pressure must be greater than the pressure P_1 which acts at the surface of contact of the two innermost elementary cylinders; and the pressures at the different surfaces of contact must diminish successively, P_1 being greater than P_2, P_2 greater than P_3, and so on; for the reason that each of these pressures is balanced by the contractile force of only that part of the compound cylinder which is outside of it.

We will first consider a compound cylinder composed of two elementary cylinders so assembled that each reaches the limit of its elastic strength when the internal pressure P_o acts.

Then, since the outer cylinder is at its elastic limit of strain under the sole action of an internal pressure P_1, we have, applying (20),

$$P_1 = \frac{3(R_2^2 - R_1^2)}{4R_2^2 + 2R_1^2}\,\theta_1 \qquad (29)$$

And, since the inner cylinder is at its elastic limit of strain under the joint action of an internal pressure P_o and an external pressure

P_1, of which pressures P_o is the greater, we have, applying (27) and (28),

either
$$P_o(\rho) = \frac{3(R_1^2 - R_o^2)\rho_o + 2P_1 R_1^2}{4R_1^2 - 2R_o^2} \tag{30}$$

or
$$P_o(\theta) = \frac{3(R_1^2 - R_o^2)\theta_o + 6P_1 R_1^2}{4R_1^2 + 2R_o^2} \tag{31}$$

of which (30) gives the value of P_o which will bring the inner surface to its elastic limit of strain by *radial compression,* while (31) gives the value of P_o which will bring the inner surface to its elastic limit of strain by *circumferential extension.* The least of these two values of P_o is the true value of the maximum allowable internal pressure, but, since which will be the least depends upon the values of P_1, R_o and R_1, we have to express both values, and we therefore distinguish between them as shown.

40. Having ascertained what maximum internal pressure our assumed compound cylinder will safely withstand, we have next to determine its condition when the internal pressure is removed, for no part of it must be overstrained either in the state of action or in the state of rest.

The state of rest differs from the state of action solely in the cessation of P_o; this must reduce P_1, and consequently the outer cylinder, which is subjected to no other pressure than P_1, must be under less strain after the removal of P_o than while it acts; the inner cylinder, however, while under a less external pressure, is no longer supported by P_o and so may be under greater strain in the state of rest than it was in the state of action. To determine whether this be so, we must find the value of the external pressure to which the inner cylinder is subjected after P_o has been removed.

Putting $r = R_1$ in (13), we obtain for the value of the circumferential strain at the outer surface of the inner cylinder (R_n and P_n becoming R_1 and P_1 in this case),

$$e_t(R_1) = \frac{1}{E}\left[\frac{6P_o R_o^2 - P_1(4R_o^2 + 2R_1^2)}{3(R_1^2 - R_o^2)}\right] \tag{32}$$

Also, remembering that the radii of the outer cylinder are R_1 and R_2, and that it is subjected only to an internal pressure P_1, we

obtain for the value of the circumferential strain at the inner surface of the outer cylinder

$$e_t(R_1') = \frac{1}{E}\left[\frac{P_1(2R_1^2 + 4R_2^2)}{3(R_2^2 - R_1^2)}\right] \tag{33}$$

These equations, giving the strains caused by the pressures P_o and P_1, will also give the changes of strain resulting from simultaneous changes of the pressures (p_o and p_1). But the surfaces of contact of the elementary cylinders must contract and expand together, and so the change of circumferential strain at the outer surface of the inner cylinder must equal that which simultaneously occurs at the inner surface of the cylinder embracing it. Hence, substituting p_o for P_o and p_1 for P_1 in the second numbers of (32) and (33), and equating them, we obtain the following relation between simultaneous changes of pressure at $r = R_o$ and $r = R_1$:

$$\frac{6\,p_o R_o^2 - p_1(4R_o^2 + 2R_1^2)}{3(R_1^2 - R_o^2)} = \frac{p_1(2R_1^2 + 4R_2^2)}{3(R_2^2 - R_1^2)}$$

$$3p_o R_o^2 (R_2^2 - R_1^2) = 3p_1 R_1^2 (R_2^2 - R_o^2)$$

$$p_1 = \frac{R_o^2 (R_2^2 - R_1^2)}{R_1^2 (R_2^2 - R_o^2)} p_o \tag{34}$$

Any change of pressure (p_o) at the inner surface, where $r = R_o$, will cause the change of pressure (p_1) at the surface of contact, where $r = R_1$, given by (34); and, vice versa, any change p_1 will cause the change p_o, given by (34). Therefore, putting $p_o = -P_o$ in (34) we have the change in P_1 which results from the suppression of the internal pressure P_o, and so $\overline{P}_1 = P_1 - \frac{R_o^2 (R_2^2 - R_1^2)}{R_1^2 (R_2^2 - R_o^2)} P_o$ is the external pressure to which the inner cylinder is subjected in the state of rest, and this must not exceed $\frac{R_1^2 - R_o^2}{2R_1^2} p_o$, which has been shown in **31** to be the greatest external pressure which, acting alone on the cylinder, is allowable.

41. The Shrinkage.—The excesses of the exterior diameters of the elementary cylinders, before assemblage, over the interior diameters of the cylinders which are to embrace them are called the *shrinkages*, and are designated by S_1, S_2, S_3 etc., S_1 being the shrinkage of the cylinder whose interior radius is R_1, S_2 that of the cylinder whose

interior radius is R_2 etc.* The differences of diameter per unit of diameter, $\frac{S_1}{2R_1}$, $\frac{S_2}{2R_2}$, $\frac{S_3}{2R_3}$ etc., are called the *relative shrinkages*, and are designated by ϕ_1, ϕ_2, ϕ_3 etc.

Referring to Figure 9, Oa and Ob represent the inner and outer radii of the inner of two elementary cylinders, and Ob' and Oc the inner and outer radii of the outer one, before assembling, so that $2b'b = S_1$ is the shrinkage; while OA, OB and OC represent the inner radius (R_o), the radius of the surface of contact (R_1) and the outer radius (R_2) after assemblage. When the internal pressure P_o acts, the compound cylinder is expanded, the three radii becoming OA', OB' and OC', respectively, and, by hypothesis, in

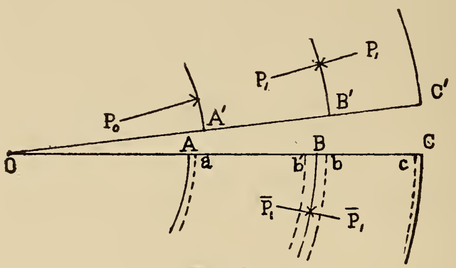


FIG. 9.

this state the inner surface of the outer cylinder is under the circumferential true stress θ_1; *i. e.*, its circumferential strain is $\frac{\theta_1}{E}$. But the change of the inner radius of the outer cylinder from its free state to the state of action is $OB' - Ob'$; therefore $OB' - Ob' = \frac{R_1\theta_1}{E}$. And the change of the outer radius of the inner cylinder from its free state to the state of action is $OB' - Ob$, and this, by (32), is $R_1\, e_t(R_1) = \frac{R_1}{E}\left[\frac{6P_o R_o^2 - P_1(4R_o^2 + 2R_1^2)}{3(R_1^2 - R_o^2)}\right]$. Hence $S_1 = 2b'b = 2[OB' - Ob' - (OB' - Ob)]$ is given by

$$S_1 = \frac{2R_1}{E}\left[\theta_1 - \frac{6P_o R_o^2 - P_1(4R_o^2 + 2R_1^2)}{3(R_1^2 - R_o^2)}\right] \qquad (35)$$

42. The formulæ which we have deduced for this case of a com-

* The shrinkages are so small in comparison with the radii that it is unnecessary to distinguish $R_1 \pm S_1$ from R_1, $R_2 \pm S_2$ from R_2 etc., in the various formulæ.

pound cylinder composed of but two elementary cylinders are grouped together in (36).

$$
\left.
\begin{array}{lll}
(a) & P_1 & = \dfrac{3\,(R_2^2 - R_1^2)}{4R_2^2 + 2R_1^2}\,\theta_1 \\[2ex]
(b) & P_o(\theta) & = \dfrac{3\,(R_1^2 - R_o^2)\,\theta_o + 6P_1\,R_1^2}{4R_1^2 + 2R_o^2} \\[2ex]
(b') & P_o(\rho) & = \dfrac{3\,(R_1^2 - R_o^2)\,\rho_o + 2P_1\,R_1^2}{4R_1^2 - 2R_o^2} \\[2ex]
(c) & \overline{P}_1 & \left(= P_1 - \dfrac{R_o^2\,(R_2^2 - R_1^2)}{R_1^2\,(R_2^2 - R_o^2)}\,P_o\right) < \dfrac{R_1^2 - R_o^2}{2R_1^2}\,\rho_o \\[2ex]
(d) & S_1 & = \dfrac{2R_1}{E}\left[\theta_1 + \dfrac{P_1\,(4R_o^2 + 2R_1^2) - 6P_o\,R_o^2}{3\,(R_1^2 - R_o^2)}\right]
\end{array}
\right\} \quad (36)
$$

To apply these formulæ, calculate P_1 and the two values of P_o by (a), (b) and (b′), using for θ_1, θ_o and ρ_o the elastic limits of the material as determined in the testing machine; then, with P_1 and the least of the two values of P_o, determine whether the condition required by (c) is fulfilled; if it is, calculate S_1 with the same values of P_1 and P_o; if it is not, find new values of P_1 and P_o, using the same values of θ_o and ρ_o but a value of θ_1 sufficiently less than the first value assigned it to cause the condition of (c) to be met.

43. As an example, we will determine the strength of a compound cylinder of steel for which $R_o = 3$ in., $R_1 = 5$ in., $R_2 = 8$ in., $\theta_1 = 24$ tons per sq. in., and $\theta_o = \rho_o = 18$ tons per sq. in.

$$P_1 = \frac{3\,(64 - 25)}{256 + 50} \times 24 = 9.18$$

$$P_o(\theta) = \frac{3\,(25 - 9) \times 18 + 6 \times 25 \times 9.18}{100 + 18} = 18.99$$

$$P_o(\rho) = \frac{3\,(25 - 9) \times 18 + 2 \times 25 \times 9.18}{100 - 18} = 16.13$$

An internal pressure of 16.13 tons per sq. in. will bring the radial strain of the inner surface to the elastic limit, and so this is the greatest safe pressure, although the circumferential strain does not reach the elastic limit unless the internal pressure is raised to 18.99

tons per sq. in. We therefore proceed to see if the condition of equation (c) is met with the values $P_1 = 9.18$, $P_o = 16.13$.

$$9.18 - \frac{9\,(64 - 25)}{25\,(64 - \ \ 9)} \times 16.13 < \frac{25 - 9}{50} \times 18$$

$$9.18 - 4.12 < 5.76$$
$$5.06 < 5.76$$

The external pressure on the inner cylinder in the state of rest is 5.06 tons per sq. in., while it is capable of withstanding 5.76 tons per sq. in. Therefore the values $P_1 = 9.18$ and $P_o = 16.13$ are allowable, and we proceed to determine the shrinkage.

$$S_1 = \frac{10}{13000}\left[\,24 + \frac{9.18\,(36 + 50) - 6 \times 16.13 \times 9}{3\,(25 - 9)}\right]$$

$$S_1 = \frac{10}{13000}\left[24 - 1.699\right] = .01715$$

The inner diameter of the outer cylinder must be bored to a diameter .01715 inches less than the outer diameter of the inner cylinder, and, if assembled with this shrinkage, the compound cylinder can be safely subjected to the internal pressure 16.13 tons per sq. in.

44. If the shrinkage used in assembling the compound cylinder be known, the resulting strains and elastic strength are determined as follows:

As shown in **41** and illustrated by Figure 9, the shrinkage is the sum of the contraction of the inner diameter of the outer cylinder and the expansion of the outer diameter of the inner cylinder which would result from disassembling them. In other words, the relative shrinkage is given by $\phi_1 = e_t(R_1') - e_t(R_1)$, in which $e_t(R_1')$ and $e_t(R_1)$ are the circumferential strains at the two surfaces of contact which the pressure between them after assemblage (in this case $\overline{P}_1$) causes. The values of these two strains being obtained by applying (13), we have

$$\frac{1}{E}\left(\frac{\overline{P}_1(2R_1^2 + 4R_2^2)}{3\,(R_2^2 - R_1^2)}\right) + \frac{1}{E}\left(\frac{\overline{P}_1(4R_o^2 + 2R_1^2)}{3\,(R_1^2 - R_o^2)}\right) = \phi_1$$

$$\overline{P}_1 = E\,\frac{(R_1^2 - R_o^2)\,(R_2^2 - R_1^2)}{2R_1^2\,(R_2^2 - R_o^2)}\,\phi_1 \qquad (37)$$

This equation (37) gives the value of the pressure at the surface of contact caused by placing a cylinder of radii R_1 and R_2 over a

cylinder of radii R_o and R_1 with the relative shrinkage ϕ_1, and the resulting circumferential strain at R_o being $-\frac{1}{E}\frac{2\overline{P}_1 R_1^2}{R_1^2-R_o^2}$, we have

$$\bar{e}_t(R_o) = -\frac{R_2^2-R_1^2}{R_2^2-R_o^2}\phi_1 \tag{38}$$

by which the relative compression of the bore of the inner cylinder caused by superposing the outer cylinder with the relative shrinkage ϕ_1 may be computed.

Since the only stress at the inner surface in the state of rest is the circumferential compression, the radial strain is one-third the circumferential strain given by (38).

EXAMPLES IV.

(1) Given $R_o = 1.80''$, $R_1 = 2.85''$, $R_2 = 4.50''$, $\theta_o = \rho_o = 18.75$ tons, $\theta_1 = \rho_1 = 21.50$ tons; find $P_o(\theta)$, $P_o(\rho)$ and S_1; also the compression at R_o in the state of rest.

$P_o(\theta) = 17.11$; $P_o(\rho) = 15.58$ tons.
$S_1 = .0074$ in.
$\overline{P}_1 = 3.61$; $E\bar{e}_t(R_o) = -12.02$ tons.

(2) Given $R_o = 2.85''$, $R_1 = 4.70''$, $R_2 = 7.50''$, $\theta_o = \rho_o = 18.75$ tons, $\theta_1 = \rho_1 = 21.5$ tons; find $P_o(\theta)$, $P_o(\rho)$ and S_1; also the compression at R_o in the state of rest.

$P_o(\theta) = 17.88$; $P_o(\rho) = 15.91$ tons.
$S_1 = .0130$ in.
$\overline{P}_1 = 4.03$; $E\bar{e}_t(R_o) = -12.75$ tons.

(3) Given $R_o = 4.00''$, $R_1 = 6.35''$, $R_2 = 8.04''$, $\theta_o = \rho_o = 18.5$ tons, $\theta_1 = \rho_1 = 21.0$ tons; find $P_o(\theta)$, $P_o(\rho)$ and S_1; also the compression at R_o in the state of rest.

$P_o(\theta) = 12.64$; $P_o(\rho) = 13.26$ tons.
$S_1 = .0126$ in.
$\overline{P}_1 = 2.01$; $E\bar{e}_t(R_o) = -6.67$ tons.

(4) Given $R_o = 6.00''$, $R_1 = 8.70''$, $R_2 = 10.46''$, $\theta_o = \rho_o = 18.5$ tons, $\theta_1 = \rho_1 = 21.0$ tons; find $P_o(\theta)$, $P_o(\rho)$ and S_1; also the compression at R_o in the state of rest.

$P_o(\theta) = 10.25$; $P_o(\rho) = 11.91$ tons.
$S_1 = .0148$ in.
$\overline{P}_1 = 1.37$; $E\bar{e}_t(R_o) = -5.22$ tons.

(5) Given $R_o = 4.00''$, $R_1 = 5.80''$, $R_2 = 7.14''$, $\theta_o = \rho_o = 18.5$ tons, $\theta_1 = \rho_1 = 21.0$ tons; find $P_o(\theta)$, $P_o(\rho)$ and S_1; also the compression at R_o in the state of rest.

$$P_o(\theta) = 10.60; \; P_o(\rho) = 12.19 \text{ tons.}$$
$$S_1 = .0105 \text{ in.}$$
$$\overline{P}_1 = 1.53; \; E\overline{e}_t(R_o) = -5.83 \text{ tons.}$$

(6) Given $R_o = 4.00''$, $R_1 = 5.80''$, $R_2 = 7.14''$, if the shrinkage was $S_1 = .0105$, what is the pressure at the surface of contact and what is the compression of the bore (at R_o) in the state of rest? (Compare result with answers to Example (5).)

$$\overline{P}_1 = 1.53; \; E\overline{e}_t(R_o) = -5.83 \text{ tons.}$$

CHAPTER V.

THE ELASTIC STRENGTH OF COMPOUND CYLINDERS.— CONTINUED.

45. The true stresses, circumferential and radial, at the inner and outer surfaces of each of the elementary cylinders are readily calculated by (13) and (14), which, when applied to the case of a compound cylinder of two parts, become

Circumferential True Stresses.

$$\left.\begin{aligned} Ee_t(R_o) &= \frac{P_o(2R_o^2+4R_1^2)-6P_1R_1^2}{3(R_1^2-R_o^2)} \\ Ee_t(R_1) &= \frac{6P_oR_o^2-P_1(4R_o^2+2R_1^2)}{3(R_1^2-R_o^2)} \\ Ee_t(R_1') &= \frac{P_1(2R_1^2+4R_2^2)}{3(R_2^2-R_1^2)} \\ Ee_t(R_2) &= \frac{2P_1R_1^2}{R_2^2-R_1^2} \end{aligned}\right\} \quad (39)$$

Radial True Stresses.

$$(40) \quad \left\{\begin{aligned} Ee_p(R_o) &= \frac{2P_1R_1^2+P_o(2R_o^2-4R_1^2)}{3(R_1^2-R_o^2)} \\ Ee_p(R_1) &= \frac{P_1(4R_o^2-2R_1^2)-2P_oR_o^2}{3(R_1^2-R_o^2)} \\ Ee_p(R_1') &= \frac{P_1(2R_1^2-4R_2^2)}{3(R_2^2-R_1^2)} \\ Ee_p(R_2) &= \frac{-2P_1R_1^2}{3(R_2^2-R_1^2)} \end{aligned}\right.$$

in which, for the state of action, P_o and P_1 have the values used in calculating the shrinkage, and, for the state of rest, P_o is zero and P_1 is the pressure at the surface of contact when $P_o=0(\overline{P}_1)$.

Applying (39) and (40) to the example worked out in **43**, for which $P_o=16.13$, $P_1=9.18$ and $\overline{P}_1=5.06$, we obtain the results illustrated in Figure 10, the right-hand side of which represents circumferential and the left-hand side radial true stresses, full lines indicating the state of action and dotted lines the state of rest.

It will be seen that in the state of action both cylinders are at the elastic limit of strain, the inner one radially and the outer one circumferentially.

46. The fact that the greater the value of P_o used in calculating the shrinkage the less the shrinkage and consequently the less the stresses in the state of rest, suggests an investigation of the results of always using $P_o(\theta)$ in (36d) instead of using $P_o(\rho)$ when it is the smaller of the two values of P_o.

In the example of **43** the shrinkage found by using $P_o(\rho)=16.13$ tons was 0.01715″; if we had used $P_o(\theta)=18.99$ tons, we would have found the shrinkage to be 0.01468″, or nearly 0.0025″ less. The

true stresses in the states of action and of rest have been computed for the greater shrinkage; we will now determine their values under the same conditions ($P_o = 16.13$ tons and $P_o = 0$), supposing the reduced shrinkage to be used.

With the reduced shrinkage the value $P_1 = 9.18$ corresponds to $P_o = 18.99$, and so we have first to find the change in P_1 which results from reducing P_o from 18.99 to 16.13; this by (34) is -0.73, making the value of P_1 for our assumed state of action $9.18 - 0.73 = 8.45$. Substituting the values $P_o = 16.13$ and $P_1 = 8.45$ in (39) and (40), we obtain the values of the true stresses in the state of action. For the state of rest we find $\overline{P_1} = 4.33$ by

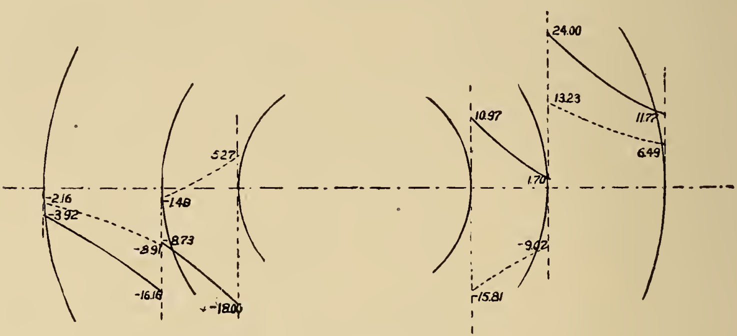


FIG. 10.

Radial True Stresses.
———— State of action.
– – – – – " " rest.

Circumferential True Stresses.
———— State of action.
– – – – – " " rest.

(36c), getting the same result, of course, whether we use $P_o = 18.99$ and $P_1 = 9.18$ or $P_o = 16.13$ and $P_1 = 8.45$; then, putting $P_o = 0$ and $P_1 = \overline{P_1} = 4.33$ in (39) and (40), we get the values of the true stresses in the state of rest.

The following table gives, side by side, the true stresses resulting from the use of the full and the reduced shrinkages:

		Circumferential Stress. Full Shrinkage.	*Circumferential Stress.* Reduced Shrinkage.	*Radial Stress.* Full Shrinkage.	*Radial Stress.* Reduced Shrinkage.
State of Action. $P_o = 16.13$ tons.	Inner cylinder, inner surface	+10.97	+13.25	−18.00	−18.76
	" " outer "	+ 1.70	+ 3.01	− 8.73	− 8.52
	Outer " inner "	+24.00	+22.09	−16.16	−14.88
	" " outer "	+11.77	+10.83	− 3.92	− 3.61
State of Rest.	Inner cylinder, inner surface	−15.81	−13.53	+ 5.27	+ 4.51
	" " outer "	− 9.07	− 7.76	− 1.48	− 1.26
	Outer " inner "	+13.23	+11.33	− 8.91	− 7.62
	" " outer "	+ 6.49	+ 5.55	− 2.16	− 1.85

47. It will be seen that the reduced shrinkage, given by adopting $P_o(\theta)$ instead of $P_o(\rho)$ as the value of P_o, results in a slight loss of elastic strength,* since the internal pressure (16.13 tons) which with full shrinkage just compressed the inner surface to its elastic limit of strain radially, with the reduced shrinkage compresses that surface slightly beyond its elastic limit. As an offset to this, the smaller shrinkage considerably reduces all the stresses in the state of rest, and those of the outer cylinder in the state of action. Moreover, there is reason to suppose that the elastic strength to resist radial compression in the case of a cylinder wall confined by an outer cylinder is greater than would be indicated by the elastic limit of compression of specimens of its material, so that the value of $P_o(\rho)$ may probably be exceeded without producing any permanent set. At all events, it is not radial compression, but circumferential extension, an excessive value of which will cause enlargement and ultimately rupture, and we are therefore adopting a measure of safety when we adjust the shrinkage so as to cause the elementary cylinders to reach their elastic limits of circumferential strain simultaneously, even though it be under a pressure greater than that which will cause the inner one of them to reach its elastic limit of radial strain.

For these reasons the Ordnance Departments of the United States Army and Navy have adopted the practice of disregarding the values of $P_o(\rho)$ and determining the shrinkages for the superposed cylinders of their built-up steel guns by using the values of $P_o(\theta)$.

We will follow the same method, using $P_o(\theta)$ for computing shrinkages, but still regarding $P_o(\rho)$, when it is less than $P_o(\theta)$, as the upper limit of safe internal pressure.

48. In **40**, by equating the simultaneous changes of circumferential strain of the two surfaces in contact at R_1, we found the relation (34) between simultaneous changes of P_o and P_1 in the case of a compound cylinder composed of two elementary cylinders. The same relation might as readily have been found from the consideration that, within the elastic limit, the stresses and strains

* With the reduced shrinkage the internal pressure which will bring the inner surface to its elastic limit of radial strain is given by

$$P_o = \frac{R_2^2 - R_o^2}{R_1^2 - R_o^2} \cdot \frac{3\,(R_1^2 - R_o^2)\,\rho_o + 2\overline{P}_1 R_1^2}{4R_2^2 - 2R_o^2},$$

the value of which for the example of 43 is 15.62 tons.

resulting from the application of any force are independent of prior stresses and strains, so that the effect of an internal pressure is exactly the same upon a compound cylinder as it would be upon a simple cylinder of the same dimensions. Thus, putting $P_n = 0$ and substituting R_2 for R_n in (12), we obtain for the pressure at any point in a homogeneous cylinder of radii R_o and R_2 under the sole pressure P_o,

$$P(r) = \frac{P_o R_o^2}{R_2^2 - R_o^2}\left(\frac{R_2^2}{r^2} - 1\right) \tag{41}$$

and, making $r = R_1$ in this, we find

$$P(R_1) = \frac{R_o^2 (R_2^2 - R_1^2)}{R_1^2 (R_2^2 - R_o^2)} P_o$$

which is the same as the relation given by (34).

49. The general principle of which the foregoing is an illustration may be stated as follows:

If any pressure be applied to a compound cylinder, the strain (or stress) at each point will be the algebraic sum of the strain (or stress) at the point before the pressure was applied and the strain (or stress) which the same pressure would cause at the corresponding point in a simple cylinder of the same dimensions as the compound one.

50. An important application of this principle shows that the maximum strength of any compound cylinder to resist internal pressure cannot exceed three-fourths the sum of the elastic limits of tension and compression of its inner elementary cylinder, regardless of the strength of its outer parts. For in the state of rest the pressure upon the inner cylinder due to the outer ones is limited to that which will compress the inner surface circumferentially to its elastic limit of compressive strain $\frac{\rho_o}{E}$; and in the state of action the internal pressure is limited to that which will extend the inner surface circumferentially to its elastic limit of tensile strain $\frac{\theta_o}{E}$; therefore the greatest allowable value of P_o is that which, acting upon a simple cylinder of the same dimensions as the compound one, would cause the circumferential strain $\frac{\rho_o + \theta_o}{E}$ at its inner

surface, and, calling the inner and outer radii R_o and R_n, the value of this greatest pressure is by (20),

$$P_o = \frac{3\,(R_n^2 - R_o^2)}{4R_n^2 + 2R_o^2}\,(\rho_o + \theta_o) \tag{42}$$

the maximum value of which, when $R_n = \infty$, is $\frac{3}{4}\,(\rho_o + \theta_o)$, or, when $\theta_o = \rho_o$, $\frac{3}{2}\theta$.

This maximum possible value of the elastic resistance will hereafter be denoted by $[P_o]$, and, since we accept the condition $\rho_o = \theta_o$, it will be written

$$[P_o] = \frac{3\,(R_n^2 - R_o^2)}{2\,R_n^2 + R_o^2}\,\theta_o. \tag{43}$$

This is the maximum possible value of $P_o(\theta)$; $P_o(\rho)$ cannot exceed ρ_o in value.

51. From the formulæ for a compound cylinder of two parts, those for the general compound cylinder (of n parts) may be directly derived, but as the case of three elementary cylinders is the commonest in gun construction, we will deduce the formulæ for that case separately, and explain how they should be used.

We begin by finding the values of the pressures in the state of action (P_2, P_1 and P_o), supposing the cylinders to have been so assembled that they reach their elastic limits of circumferential strain simultaneously.

The outer cylinder being under the sole action of the internal pressure P_2, we have from (20),

$$P_2(\theta) = \frac{3\,(R_3^2 - R_2^2)}{4\,R_3^2 + 2R_2^2}\,\theta_2 \tag{44}$$

The middle cylinder being under the external pressure P_2 and the internal pressure P_1, of which the latter is the greater, we have from (28),

$$P_1(\theta) = \frac{3\,(R_2^2 - R_1^2)\,\theta_1 + 6P_2\,R_2^2}{4R_2^2 + 2R_1^2} \tag{45}$$

And the inner cylinder being under the external pressure P_1 and the internal pressure P_o, of which the latter is the greater, we have from (28),

$$P_o(\theta) = \frac{3\,(R_1^2 - R_o^2)\,\theta_o + 6P_1\,R_1^2}{4R_1^2 + 2R_o^2} \tag{46}$$

Before adopting these values of P_2, P_1 and P_o, we must see that the shrinkages which they require will not over-compress the inner surface in the state of rest. This is most readily done by computing $[P_o]$ by (43) and comparing it with $P_o(\theta)$; if the latter be the greater, the inner surface would be compressed beyond its elastic limit of circumferential strain when in the state of rest, and so less values must be assigned to one or both the elastic limits of the outer cylinders and new values of P_2, P_1 and P_o computed. When the assumed values of θ_2, θ_1 and θ_o are such that $[P_o]$ exceeds $P_o(\theta)$, the inner cylinder will not be too much compressed, and then the values of $P_2(\theta)$, $P_1(\theta)$ and $P_o(\theta)$ given by (42), (43) and (44) may be accepted.

52. The formulæ for the shrinkages are deduced by the same method that was explained in **41.** The inner surface of the outer cylinder when in the state of action is, by hypothesis, under the circumferential strain $\frac{\theta_2}{E}$, so that its diameter is $2R_2\frac{\theta_2}{E}$ greater than when it was free (before assembling). If, then, we find the change of diameter $(2R_2 e_t(R_2))$ of the outer surface of the middle cylinder which would result from the simultaneous removal of the outer cylinder and suppression of the internal pressure P_o, the shrinkage with which the outer cylinder was assembled will evidently be given by $S_2 = 2R_2\frac{\theta_2}{E} + 2R_2 e_t(R_2)$.

By substituting R_2 for R_n, P_2 for P_n and R_2 for r in (13) we obtain the following expression for the circumferential strain at the outer surface of a cylinder of radii R_o and R_2 under internal pressure P_o and external pressure P_2:

$$e_t(R_2) = \frac{1}{E}\left[\frac{6P_o R_o^2 - P_2(4R_o^2 + 2R_2^2)}{3(R_2^2 - R_o^2)}\right] \tag{47}$$

But by the principle laid down in **49** the same expression gives the change of strain which the application of the same pressures will cause in a compound cylinder of the same dimensions. Therefore, putting $-P_o$ for P_o and $-P_2$ for P_2 in (47), we obtain the change of circumferential strain at R_2 due to suppressing P_o and P_2, and this multiplied by $2R_2$ will be the change of diameter. Consequently the shrinkage of the outer cylinder is given by

$$S_2 = \frac{2R_2}{E}\left[\theta_2 + \frac{P_2(4R_o^2 + 2R_2^2) - 6P_o R_o^2}{3(R_2^2 - R_o^2)}\right] \tag{48}$$

Similarly the change of circumferential strain at the outer surface of the inner cylinder due to removing the two outer cylinders (*i. e.,* suppressing P_1) and simultaneously suppressing P_o is found to be

$$e_t(R_1) = \frac{1}{E}\left[\frac{P_1(4R_o^2 + 2R_1^2) - 6P_o R_o^2}{3(R_1^2 - R_o^2)}\right] \tag{49}$$

and so the shrinkage of the middle cylinder is

$$S_1 = \frac{2R_1}{E}\left[\theta_1 + \frac{P_1(4R_o^2 + 2R_1^2) - 6P_o R_o^2}{3(R_1^2 - R_o^2)}\right] \tag{50}$$

52. We have, finally, to determine the elastic strength to resist internal pressure of the system thus assembled. We know that $P_o(\theta)$ is the pressure which will bring its elementary cylinders simultaneously to their assumed elastic limits of circumferential strain, but a less pressure may bring one or more of them to the elastic limit of radial strain, and, if so, this latter pressure, and not $P_o(\theta)$, should be taken as the maximum safe pressure.

The outer cylinder being under internal pressure only, $P_2(\theta)$ is always less than $P_2(\rho)$, as explained in **29.** Applying (27) to the middle and inner cylinders, we obtain the following values for the respective internal pressures which will bring them to their elastic limits of radial strain:

$$P_1(\rho) = \frac{3(R_2^2 - R_1^2)\rho_1 + 2P_2 R_2^2}{4R_2^2 - 2R_1^2} \tag{51}$$

$$P_o(\rho) = \frac{3(R_1^2 - R_o^2)\rho_o + 2P_1 R_1^2}{4R_1^2 - 2R_o^2} \tag{52}$$

If $P_1(\rho)$ given by (51) is less than the value of $P_1(\theta)$ used in computing the shrinkages, then the former is to be used for P_1 in (52) instead of the latter, and if $P_o(\rho)$ given by (52) is less than the value of $P_o(\theta)$ used in computing the shrinkages, it, and not $P_o(\theta)$, is the maximum safe pressure. That is, with $P_o(\rho) < P_o(\theta)$, the former would be a safe pressure if suitable shrinkages were assigned, but since, for good reasons, we adopt shrinkages based upon the values of $P_1(\theta)$ and $P_o(\theta)$, the actual maximum safe

pressure is somewhat less than $P_o(\rho)$. We will call the true maximum safe pressure P_o, thus distinguishing it from $P_o(\rho)$ and $P_o(\theta)$; its value, when it does not equal $P_o(\theta)$, is found as follows:

The pressures in the state of rest are given by (53) and (54), the negative part of each value being the change of pressure due to the suppression of $P_o(\theta)$:

$$\overline{P}_2 = P_2(\theta) - \frac{R_o^2\,(R_3^2 - R_2^2)}{R_2^2\,(R_3^2 - R_o^2)}\,P_o(\theta) \tag{53}$$

$$\overline{P}_1 = P_1(\theta) - \frac{R_o^2\,(R_3^2 - R_1^2)}{R_1^2\,(R_3^2 - R_o^2)}\,P_o(\theta) \tag{54}$$

Then by (14) the radial strain at the inner surface of the inner cylinder, in the state of rest, is $\frac{1}{E}\,\frac{2\overline{P}_1 R_1^2}{3\,(R_1^2 - R_o^2)}$, and the internal pressure which will change this radial strain to $-\frac{\rho_o}{E}$, *i. e.*, which will bring the inner surface to its elastic limit of compression radially, is, by the principle of **49**,

$$P_o = \frac{3\,(R_3^2 - R_o^2)}{2\,R_o^2 - 4R_3^2}\left(-\frac{2\overline{P}_1 R_1^2}{3\,(R_1^2 - R_o^2)} - \rho_o\right)$$

$$P_o = \frac{R_3^2 - R_o^2}{R_1^2 - R_o^2}\cdot\frac{3\,(R_1^2 - R_o^2)\,\rho_o + 2\overline{P}_1 R_1^2}{4R_3^2 - 2R_o^2} \tag{55}$$

The same method applied to the middle cylinder, which in the state of rest is acted on by P_2 externally and P_1 internally, would determine the internal pressure which would bring its inner surface to the elastic limit of compression radially,* but this pressure will practically always be greater than that given by (55), and, accordingly, (55) gives the true elastic strength of the system.

* The formula is

$$P_o = \frac{R_1^2\,(R_3^2 - R_o^2)}{R_o^2\,(4R_3^2 - 2R_1^2)\,(R_2^2 - R_1^2)}\left[3\,(R_2^2 - R_1^2)\,\rho_1 + 2\overline{P}_2 R_2^2 - \overline{P}_1\,(4R_2^2 - 2R_1^2)\right]$$

54. The formulæ for the case of a compound cylinder composed of three elementary cylinders are grouped together in (56):

$$\left.\begin{aligned}
&(a)\quad P_2(\theta) = \frac{3\,(R_3^2 - R_2^2)\,\theta_2}{4\,R_3^2 + 2R_2^2}\\
&(b)\quad P_1(\theta) = \frac{3\,(R_2^2 - R_1^2)\,\theta_1 + 6P_2\,R_2^2}{4R_2^2 + 2R_1^2}\\
&(c)\quad P_o(\theta) = \frac{3\,(R_1^2 - R_o^2)\,\theta_o + 6P_1\,R_1^2}{4R_1^2 + 2R_o^2}\\
&(d)\quad P_o(\theta) < \frac{3\,(R_3^2 - R_o^2)\,(\theta_o + \rho_o)}{4R_3^2 + 2R_o^2}\\
&(e)\quad P_1(\rho) = \frac{3\,(R_2^2 - R_1^2)\,\rho_1 + 2P_2\,R_2^2}{4R_2^2 - 2R_1^2}\\
&(f)\quad P_o(\rho) = \frac{3\,(R_1^2 - R_o^2)\,\rho_o + 2P_1\,R_1^2}{4R_1^2 - 2R_o^2}\\
&(g)\quad S_2 = \frac{2R_2}{E}\left[\theta_2 + \frac{P_2(\theta)\,(4R_o^2 + 2R_2^2) - 6P_o(\theta)\,R_o^2}{3\,(R_2^2 - R_o^2)}\right]\\
&(h)\quad S_1 = \frac{2R_1}{E}\left[\theta_1 + \frac{P_1(\theta)\,(4R_o^2 + 2R_1^2) - 6P_o(\theta)\,R_o^2}{3\,(R_1^2 - R_o^2)}\right]\\
&(i)\quad \overline{P}_2 = P_2(\theta) - \frac{R_o^2\,(R_3^2 - R_2^2)}{R_2^2\,(R_3^2 - R_o^2)}\,P_o(\theta)\\
&(j)\quad \overline{P}_1 = P_1(\theta) - \frac{R_o^2\,(R_3^2 - R_1^2)}{R_1^2\,(R_3^2 - R_o^2)}\,P_o(\theta)\\
&(k)\quad P_o = \frac{R_3^2 - R_o^2}{R_1^2 - R_o^2}\cdot\frac{3\,(R_1^2 - R_o^2)\,\rho_o + 2\overline{P}_1\,R_1^2}{4R_3^2 - 2R_o^2}
\end{aligned}\right\}\quad(56)$$

The method of procedure is as follows:

(1) Calculate $P_2(\theta)$, $P_1(\theta)$ and $P_o(\theta)$ by (a), (b) and (c), using for θ_2, θ_1 and θ_o the elastic limits of the materials as determined in a testing machine.

(2) See if the condition (d) is fulfilled. If it is not, find new values of $P_2(\theta)$, $P_1(\theta)$ and $P_o(\theta)$, using values of θ_2 and θ_1 (one or both) sufficiently less than their true values to cause the condition (d) to be met.

(3) Calculate the shrinkages by (g) and (h), using the values of θ_2, θ_1, $P_2(\theta)$, $P_1(\theta)$ and $P_o(\theta)$ which satisfied (d).

(4) Calculate $P_1(\rho)$ and $P_o(\rho)$ by (e) and (f), using for ρ_o

and ρ_1 the true elastic limits of the materials, and for P_1 in (f) putting whichever is least, $P_1(\rho)$ or the value of $P_1(\theta)$ calculated with the true values of θ_2 and θ_1.

(5) If $P_o(\rho)$ is greater than the value of $P_o(\theta)$ used in computing the shrinkages, the latter is the true measure of the elastic strength of the system; if it be less, then P_o, calculated by (k), is the true measure.

55. To find the state of strain at the inner surface (at R_o) caused by superposing the two outer cylinders with relative shrinkages, respectively ϕ_1 and ϕ_2, we have only to apply (38) to this case, the strain resulting from the compressive action of both outer cylinders being merely the sum of the strains caused by their actions considered separately. Thus we have

$$\bar{e}_t(R_o) = -\frac{R_2^2 - R_1^2}{R_2^2 - R_o^2}\phi_1 - \frac{R_3^2 - R_2^2}{R_3^2 - R_o^2}\phi_2 \qquad (57)$$

Moreover, the radial strain at the inner surface in the state of rest $(\bar{e}_p(R_o))$ will be one-third the circumferential strain given by (57).

EXAMPLES V.

(1) Given $R_o = 7.0''$, $R_1 = 9.5''$, $R_2 = 15.0''$, $R_3 = 21.0''$; if $\theta_o = \rho_o = 20.0$ tons, what is the greatest possible value of the internal pressure which can be withstood elastically? If $\theta_o = 20.0$, $\theta_1 = 21.4$ and $\theta_2 = 22.3$ tons, find $P_o(\theta)$, S_1 and S_2. What is the true elastic strength after assemblage with the shrinkages based on the value of $P_o(\theta)$?

$[P_o] = 25.26$; $P_o(\theta) = 24.46$ tons.
$S_1 = .0183$; $S_2 = .0386$.
$\overline{P}_1 = 4.28$; $P_o = 18.52$ tons.

(2) Given $R_o = 5.0''$, $R_1 = 9.5''$, $R_2 = 15.0''$, $R_3 = 19.0''$, $\theta_o = \rho_o = 20.0$ tons; what is the limiting value for the internal pressure? If $\theta_o = 20.0$, $\theta_1 = 21.0$, and $\theta_2 = 24$ tons, find $P_o(\theta)$. If assembled with the shrinkages corresponding to the value of $P_o(\theta)$, what would be the compression at R_o in the state of rest?

$[P_o] = 26.99$; $P_o(\theta) = 28.39$ tons.
22.06 tons.

(3) Given $R_o = 6.0''$, $R_1 = 10.3''$, $R_2 = 15.0''$, $R_3 = 17.7''$, $\theta_o = 17.5$ tons, $\theta_1 = 22.0$ tons, $\theta_2 = 22.0$ tons; find $[P_o]$, $P_o(\theta)$, S_1 and S_2.

$[P_o] = 21.97$; $P_o(\theta) = 21.79$ tons.
$S_1 = .0296$; $S_2 = .0400$ in.

(4) Given $R_o = 4.75''$, $R_1 = 7.50''$, $R_2 = 11.375''$, $R_3 = 14.375''$, $\theta_o = 16$ tons, $\theta_1 = 17$ tons, $\theta_2 = 22.2$ tons; find $P_o(\theta)$, S_1 and S_2.

$P_o(\theta) = 20.68$ tons; $S_1 = .0149$; $S_2 = .0327$ in.

(5) Given $R_o = 6.0''$, $R_1 = 11.0''$, $R_2 = 17.0''$, $R_3 = 21.0''$, $\theta_o = 18.0$, $\theta_1 = 19.0$, $\theta_2 = 21$ tons; find $[P_o]$, $P_o(\theta)$, S_1, S_2 and P_o.

$[P_o] = 23.82$; $P_o(\theta) = 23.82$ tons.
$S_1 = .0287$; $S_2 = .0476$ in.
$\overline{P}_1 = 6.32$; $P_o = 17.23$ tons.

(6) Given $R_o = 6.0''$, $R_1 = 11.0''$, $R_2 = 17.0''$ and $R_3 = 21.0''$, if the shrinkages were $S_1 = .0287$ and $S_2 = .0476$, find the circumferential and radial true stresses at the inner surface (at R_o) in the state of rest. Then, by the principle of **49**, find the internal pressures which will strain the inner surface to the elastic limit (18 tons) first radially and second circumferentially. (Compare results with answers to example (5).)

$E\bar{e}_t(R_o) = 18.09$; $E\bar{e}_p(R_o) = 6.03$ tons.
$P_o(\theta) = 23.87$; $P_o(\rho) = 17.25$ tons.

CHAPTER VI.

APPLICATIONS TO BUILT-UP GUNS.

56. The modern gun is essentially a compound cylinder, but, being constructed to withstand an internal pressure which diminishes from the breech end to the muzzle, the number of layers and the exterior dimensions are correspondingly decreased for economy of weight, making it necessary to divide the whole length into a number of sections for each of which a separate computation of the elastic strength and shrinkages must be made. In United States guns the inner layer, in which are formed the chamber and bore proper, is called the *tube;* the second layer consists of a *jacket,* in which the breech block is housed, and *chase hoops,* which extend from the front end of the jacket nearly or quite to the muzzle; over that part of the bore in which the maximum powder pressure acts a third and sometimes a fourth layer of *hoops* is placed. With increase of knowledge and of facilities larger and larger steel forgings of assured good quality have become available, and the number of separate parts constituting a built-up gun has tended to diminish, so that at the present time the outer layers, as well as the tube, are sometimes made in one piece.

In one particular, however, there is an important difference between a gun and the compound cylinders with free ends which we have thus far considered; in the latter there is no longitudinal stress, while in a gun the internal pressure, acting upon the breech block as well as upon the cylinder walls, gives rise to a longitudinal stress of very considerable intensity.

57. The Longitudinal Stress.—If we consider a gun recoiling freely under the action of the powder pressure on the bottom of its bore, we see that the total longitudinal stress on any cross-section of the gun must equal the product of the acceleration by the mass forward of the section, so that the said stress diminishes rapidly as we go forward from the front thread of the screw box, where it is a maximum. When recoil is resisted by a brake of any kind, the acceleration is reduced and so, to the same extent, is the longitudinal stress on all cross-sections forward of the point of attach-

ment of the brake to the gun; in rear of that point the longitudinal stress is increased by the action of the recoil brake, the increase diminishing as the cross-section through the front thread of the screw box is approached till, at that point, the total longitudinal stress is practically the same as in free recoil. When, as in most modern United States naval gun mounts, the pistons of the recoil cylinders are attached to a yoke around the breech of the gun, the longitudinal stress is diminished at all sections, its maximum value then being $\frac{M'}{M}(\pi R_0^2 P - F)$, in which M is the whole recoiling mass, M' is that part of it which is forward of the front thread of the screw box, R_0 is the radius of bore and P the maximum powder pressure, and F is the total resistance * of the recoil brake at the instant when P acts.

We do not know how the total longitudinal stress is distributed over the cross-section of the gun. It is not wholly born by the layer in which the breech block houses (the jacket in United States guns), for there is an enormous frictional resistance to the longitudinal motion of any one layer relative to the others; if it were uniformly distributed over the jacket alone, its intensity, even at the section of greatest stress, would seldom exceed 5 or 6 tons per square inch, and if, as many writers assume, it is uniformly distributed over the whole cross-section of the gun, its greatest intensity will not exceed 2 or 3 tons per square inch. Probably the latter assumption is practically true at some distance forward of the breech block and is not very far from the truth at any point forward of the gas check.

Moreover, this maximum intensity of longitudinal stress only exists for the infinitesimally small period of time during which the maximum powder pressure is maintained; during the greater part of the time in which the gun is subjected to internal pressure the longitudinal stress is very small, even at the section where it has its greatest value.

For these reasons, therefore, we are justified in applying to guns the formulæ which we have deduced for cylinders with free ends.

58. If circumferential strain alone had to be considered in the case of a compound cylinder, the greatest strength would be obtained by making the successive radii of the elementary cylinders

* This total resistance of the recoil brake, however, is never more than a small fraction of the maximum total pressure on the bottom of the bore of the gun.

increase in geometrical progression, provided their physical characteristics were the same. Thus, for the case of any one cylinder superimposed upon another, regarding $P_o(\theta)$ in (36b) as a function of R_1 (R_o and R_2 constant, and $\theta_1 = \theta_o$) and putting $\frac{dP_o(\theta)}{dR_1} = 0$, we find, after simplification, $R_1^2 = R_o R_2$, which shows that the maximum value of $P_o(\theta)$ for a given total thickness of a given material occurs when the radius of the common surface is a mean proportional between the inner and outer radii. Very nearly the same proportions will also give the greatest strength as regards radial strain.

In practice, however, other considerations govern in the proportioning of the layers of which guns are composed. In the first place the layer in which the breech block is housed, even though other layers assist it in taking the longitudinal stress, should be of sufficient cross-section to itself safely sustain that stress. Again, the thickness of the tube over the chamber should be sufficient to make relining practicable in case erosion wears away the rifling, and its thickness elsewhere should be sufficient to give ample stiffness. Finally, the necessity for keeping down the weight, which prescribes a decreasing exterior diameter towards the muzzle, and the need for avoiding sudden or great changes of the sections of the different layers, often require dimensions not otherwise desirable.

59. In assigning shrinkages for the different parts of a gun, while as a general rule the maximum attainable strength should be sought at each section, great changes of shrinkage in passing from one section to another must be avoided, as they would cause undesirable inequalities of strain. Not only should each of the parts which make up the outer layers of the gun be assembled so that the strains at its inner surface, both in the state of rest and in that of action, do not change abruptly at any point of its length, but the tube, similarly throughout its length, should be under a compression in the state of rest, and of extension in the state of action, which only gradually varies and at no point changes abruptly. Furthermore, as a rule, slack shrinkages should be preferred to excessive ones, to the end that under the action of an excessive pressure it may be the tube which gives way rather than an outer part.

60. As a simple example of the method of determining the proper shrinkages, and the elastic strength of a gun, we will consider the

case of the United States naval 5-inch B. L. R. Mark V, which is shown in Figure 11, with its curves of computed elastic strength and of strains at rest and in action.

The computations are made separately for each of the sections indicated on the drawing, but only those for the most important section, that through the chamber, will be worked out in the text, the final results of the other computations, which are obtained in exactly the same way, being merely stated. As it is always necessary to adjust the shrinkages, in accordance with the principle set forth in **59,** it is most convenient to find their values, as well as the values of the pressures in the state of action, in terms of the elastic limits of the different layers, afterwards assigning suitable values to the elastic limits, always, of course, within their true values as indicated by the testing machine.

5-inch B. L. R. Mark V.

Section I.

$$\left.\begin{array}{llll} R_o = & 3.50 & R_o^2 = & 12.25 \\ R_1 = & 5.25 & R_1^2 = & 27.56 \\ R_2 = & 8.25 & R_2^2 = & 68.06 \\ R_3 = & 10.25 & R_3^2 = & 105.06 \end{array}\right\} \quad \begin{array}{lr} 3\,(R_3^2 - R_o^2) = 278.43 \ldots\ldots \log & 2.44472 \\ 2\,R_3^2 + R_o^2 = 222.37 \ldots\ldots \text{“} & 2.34707 \\ \text{“} & \overline{0.09765} \end{array}$$

$$\left.\begin{array}{l} \theta_o = \rho_o = 20.0 \text{ tons} \\ \theta_1 = \rho_1 = 21.5 \quad \text{“} \\ \theta_2 = \rho_2 = 22.0 \quad \text{“} \end{array}\right\}* \quad \begin{array}{lr} \theta_o = \quad 20.0 \ldots\ldots \text{“} & 1.30103 \\ {[P_o]} = \quad 25.04 \ldots\ldots \text{“} & \overline{1.39868} \end{array}$$

That is, 25.04 tons is the greatest possible elastic strength, whatever the qualities of the jacket and hoop.

$$\begin{array}{lr} 3\,(R_3^2 - R_2^2) = 111.00 \ldots\ldots \log & 2.04532 \\ 4R_3^2 + 2R_2^2 = 556.36 \ldots\ldots \text{“} & 2.74536 \\ A_2 \ldots\ldots\ldots\ldots\ldots \text{“} & \overline{9.29996} \\ \theta_2 = \quad 22.0 \ldots\ldots \text{“} & 1.34242 \\ P_2(\theta) = \quad 4.389 \ldots\ldots \text{“} & \overline{.64238} \end{array}$$

$$P_2(\theta) = A_2\,\theta_2 = [9.29996]\,\theta_2$$

$$\begin{array}{lr|lr} 3\,(R_2^2 - R_1^2) = 121.50 \ldots\ldots \log & 2.08458 & 6R_2^2 = 408.36 \ldots\ldots \log & 2.61104 \\ 4R_2^2 + 2R_1^2 = 327.36 \ldots\ldots \text{“} & 2.51503 & \ldots\ldots\ldots\ldots\ldots\ldots \text{“} & 2.51503 \\ A_1 \ldots\ldots\ldots\ldots\ldots \text{“} & \overline{9.56955} & B_1 \ldots\ldots\ldots\ldots\ldots \text{“} & \overline{.09601} \\ \theta_1 = \quad 21.5 \ldots\ldots \text{“} & 1.33244 & A_2 \ldots\ldots\ldots\ldots\ldots \text{“} & 9.29996 \\ 7.980 \ldots\ldots \text{“} & \overline{.90199} & B_1\,A_2 \ldots\ldots\ldots\ldots\ldots \text{“} & \overline{9.39597} \\ & & \theta_2 = 22.0 \ldots\ldots \text{“} & 1.34242 \\ 5.475 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \ldots\ldots\ldots\ldots \text{“} & .73839 \\ P_1(\theta) = \overline{13.455} & & & \end{array}$$

$$P_1(\theta) = A_1\,\theta_1 + B_1\,A_2\,\theta_2 = [9.56955]\,\theta_1 + [9.39597]\,\theta_2$$

* These are the true elastic limits, being the least values given by any of the specimens taken respectively from the tube, jacket and hoop.

$3(R_1^2-R_o^2)=45.93$	..log 1.66210	$6R_1^2=165.36$	..log 2.21843	B_1A_2	..log 9.39597
$4R_1^2+2R_o^2=134.74$	.. " 2.12950	....	" 2.12950		
A_o	" 9.53260	B_o	" .08893	......	" .08893
$\theta_o=20.0$	.. " 1.30103	A_1	" 9.56955		
6.818..	" .83363	B_oA_1	" 9.65848	$B_oB_1A_2$..	" 9.48490
		$\theta_1=21.5$ ..	" 1.33244		
9.793		....	" .99092	$\theta_2=22.0$	" 1.34242
6.719		....	....	....	" .82732
$P_o(\theta)=23.330$					

$$P_o(\theta)=A_o\theta_o+B_oA_1\theta_1+B_oB_1A_2\theta_2=[9.53260]\,\theta_o+[9.65848]\,\theta_1+[9.48490]\,\theta_2$$

The value of $P_o(\theta)$ for the true values of the elastic limits being 23.33 tons, while $[P_o]=25.04$ tons, the inner surface is not too much compressed in the state of rest, and so we proceed to determine the shrinkages.

$6R_o^2=$ 73.50	....	....	log 1.86629		
$2R_1=$ 10.50.....	log 1.02119				
$E=13000.0$	" 4.11394				
$\frac{2R_1}{E}=.0008077$......	" 6.90725	....	" 6.90725		
$4R_o^2+2R_1^2=$ 104.12......	" 2.01753				
	" 8.92478		" 8.77354		
$3(R_1^2-R_o^2)=$ 45.93......	" 1.66210	....	" 1.66210		
	" 7.26268....	log 7.26268	" 7.11144...	log 7.11144...	log 7.11144
A_1	" 9.56955		A_o " 9.53260		
B_1A_2	....	" 9.39597 B_oA_1	....	" 9.65848	
	" 6.83223....	" 6.65865 $B_oB_1A_2$	....	....	" 9.48490
			log 6.64404	" 6.76992	" 6.59634

$+.0008077\,\theta_1 \quad + \quad .0006796\,\theta_1 \quad + \quad .0004557\,\theta_2 \quad - \quad .0004406\,\theta_o - .0005887\,\theta_1 - .0003948\,\theta_2$

$+.0008077$	$.0004557$
$+.0006796$	$-.0003948$
$+.0014873$	$+.0000609$
$-.0005887$	
$+.0008986$	

$$S_1=.0008986\,\theta_1-.0004406\,\theta_o+.0000609\,\theta_2$$

$6R_o^2=$ 73.50	....	log 1.86629		
$2R_2=$ 16.50...	log 1.21748			
$E=13000.00$...	" 4.11394			
$\frac{2R_2}{E}=.0012692$...	" 7.10354....	" 7.10354		
$4R_o^2+2R_2^2=$ 185.12...	" 2.26745			
	" 9.37099....	" 8.96983		
$3(R_2^2-R_o^2)=$ 167.43...	" 2.22383....	" 2.22383		
	" 7.14716....	" 6.74600...	log 6.74600...	log 6.74600
A_2	" 9.29996	A_o " 9.53260		
	" 6.44712 B_oA_1	....	" 9.65848	
	$B_oB_1A_2$	....	....	" 9.48490
		" 6.27860	" 6.40448	" 6.23090

$+.0012692\theta_2 \ + \ .0002800\theta_2 \ - \ .0001899\theta_o - .0002538\theta_1 - .0001702\theta_2$

$+.0002800$

$+.0015492$

$-.0001702$

$+.0013790$

$$S_2=.0013790\,\theta_2-.0001899\,\theta_o-.0002538\,\theta_1$$

Now, substituting the values 20.0, 21.5 and 22.0 for θ_o, θ_1 and θ_2, respectively, we have for the shrinkages which will cause tube, jacket and hoop to simultaneously reach their elastic limits of circumferential strain, under the internal pressure $P_o(\theta) = 23.33$ tons, $S_1 = .01185$ and $S_2 = .02108$.

In exactly the same way as shown for Section I, the values of the pressures in the state of action and the corresponding shrinkages are computed for the other sections, the results being as follows:

Section II.

$$\left.\begin{array}{l} R_o = 2.70 \\ R_1 = 5.25 \\ R_2 = 8.25 \\ R_3 = 10.25 \end{array}\right\} \begin{array}{l} P_2(\theta) = [9.29996]\,\theta_2 \\ P_1(\theta) = [9.56955]\,\theta_1 + [9.39597]\,\theta_2 \\ P_o(\theta) = [9.69769]\,\theta_o + [9.69170]\,\theta_1 + [9.51812]\,\theta_2 \\ S_1 = .0010376\,\theta_1 - .0002896\,\theta_o + .0000983\,\theta_2 \\ S_2 = .0013976\,\theta_2 - .0001508\,\theta_o - .0001488\,\theta_1 \end{array}$$

Section III.

$$\left.\begin{array}{l} R_o = 2.50 \\ R_1 = 5.25 \\ R_2 = 8.25 \\ R_3 = 9.00 \end{array}\right\} \begin{array}{l} P_2(\theta) = [8.92618]\,\theta_2 \\ P_1(\theta) = [9.56955]\,\theta_1 + [9.02219]\,\theta_2 \\ P_o(\theta) = [9.71671]\,\theta_o + [9.69899]\,\theta_1 + [9.15163]\,\theta_2 \\ S_1 = .0009470\,\theta_1 - .0002468\,\theta_o + .00005599\,\theta_2 \\ S_2 = .0012509\,\theta_2 - .0001919\,\theta_o - .0001842\,\theta_1 \end{array}$$

Section IV.

$$\left.\begin{array}{l} R_o = 2.50 \\ R_1 = 5.25 \\ R_2 = 8.00 \end{array}\right\} \begin{array}{l} P_1(\theta) = [9.54577]\,\theta_1 \\ P_o(\theta) = [9.71671]\,\theta_o + [9.67521]\,\theta_1 \\ S_1 = .0009391\,\theta_1 - .0002468\,\theta_o \end{array}$$

Section V.

$$\left.\begin{array}{l} R_o = 2.50 \\ R_1 = 5.00 \\ R_2 = 7.00 \end{array}\right\} \begin{array}{l} P_1(\theta) = [9.46639]\,\theta_1 \\ P_o(\theta) = [9.69897]\,\theta_o + [9.59133]\,\theta_1 \\ S_1 = .0008693\,\theta_1 - .0002564\,\theta_o \end{array}$$

Section VI.

$$\left.\begin{array}{l} R_o = 2.50 \\ R_1 = 5.00 \\ R_2 = 5.50 \end{array}\right\} \begin{array}{l} P_1(\theta) = [8.96428]\,\theta_1 \\ P_o(\theta) = [9.69897]\,\theta_o + [9.08922]\,\theta_1 \\ S_1 = .0008007\,\theta_1 - .0002564\,\theta_o \end{array}$$

Section VII.

$$\left.\begin{array}{l} R_o = 2.50 \\ R_1 = 5.00 \end{array}\right\} P_o(\theta) = [9.69897]\,\theta_o$$

Section VIII.

$$\left.\begin{array}{l} R_o = 2.50 \\ R_1 = 3.8735 \end{array}\right\} P_o(\theta) = [9.55930]\,\theta_o$$

Section IX.

$$\left.\begin{array}{l} R_o = 2.50 \\ R_1 = 4.50 \end{array}\right\} P_o(\theta) = [9.65244]\,\theta_o$$

61. We adopt as the shrinkages for that part of the gun which is represented by Section 1 the values $S_1 = .0120$ and $S_2 = .0210$, being (to the nearest thousandth of an inch) those which result from substituting the true values of θ_o, θ_1 and θ_2 in the expressions for S_1 and S_2.

If now we compute the shrinkages for Section II with the same values of θ_o, θ_1 and θ_2, we find $S_1 = .0165$, $S_2 = .0197$ and $P_o(\theta) =$ 27.79 tons, and as the increase of strength over the adjoining section would be valueless, while the great increase of the jacket shrinkage would cause a very undesirable inequality of strains in the state of rest, we see that it will be best to assign less values to θ_1 and θ_2 and to adopt a correspondingly less shrinkage for this section. If, on the other hand, we should adopt the same shrinkages for Section II as for Section I, an internal pressure which would bring the bore to its elastic limit would only cause a circumferential true stress of about 16.6 tons at the inner surface of the jacket, thus causing an undesirable inequality of strains in the state of action, since in the adjoining section the jacket reaches its elastic limit with the tube. We therefore compromise between the two extremes, and adopt the values $S_1 = .0130$ and $S_2 = .0200$ for part of the gun which Section II represents.

Guided by similar considerations, we assign to the shrinkages at the other sections the values stated on the drawing.

62. We might now, by means of the general values of S_1 and S_2 which we have computed for each section, find the values of θ_1 and θ_2 which, in combination with the value 20.0 for θ_o, will give the shrinkages which have been adopted, and then, with those values of θ_o, θ_1 and θ_2, calculate the elastic strength, compression of bore in the state of rest, etc. A better method, however, is to start afresh and with the given shrinkages calculate first, by (57), the circumferential and radial strains at the surface of the bore in the state of rest and then the internal pressure which will increase each of those strains to its greatest allowable value. We will do this for Section II, as an illustration.

$$\left.\begin{array}{ll} R_o = 2.70 & R_o^2 = 7.29 \\ R_1 = 5.25 & R_1^2 = 27.56 \\ R_2 = 8.25 & R_2^2 = 68.06 \\ R_3 = 10.25 & R_3^2 = 105.06 \end{array}\right\} \quad \begin{array}{l} S_1 = .0130;\ \phi_1 = \dfrac{S_1}{2R_1} = .0012381 \\ S_2 = .0200;\ \phi_2 = \dfrac{S_2}{2R_2} = .0012121 \end{array}$$

$R_2^2 - R_1^2 =$	40.5	....log 1.60746	$R_3^2 - R_2^2 = 37.0$		....log 1.56820
$\phi_1 =$	.0012381....	" 7.09275	$\phi_2 =$	.0012121....	" 7.08354
		" 8.70021			" 8.65174
$R_2^2 - R_o^2 =$	60.77	.... " 1.78369	$R_3^2 - R_o^2 = 97.77$		.... " 1.99021
	.0008251....	" 6.91652			
	.0004587....				" 6.66153
$\bar{e}_t(R_o) =$	$-$.0012838....	" 7.10850			
$E =$	13000	.... " 4.11394			
$E\bar{e}_t(R_o) =$	$-$ 16.69	.... " 1.22244			
$E\bar{e}_p(R_o) =$	+ 5.56				

The true circumferential stress at the surface of the bore in the state of rest is thus found to be — 16.69 tons, while the true radial stress is + 5.56 tons. Therefore, applying the principle laid down in **49**, the internal pressures which will, respectively, bring the inner surface to its elastic limits of strain circumferentially and radially, are found as follows:

$3(R_3^2 - R_o^2) =$	293.31..........	log 2.46733..........	log 2.46733
$\theta_o + 16.69 =$	36.69..........	" 1.56455	
$\rho_c + \ \ 5.56 =$	25.56..........		" 1.40756
		" 4.03188	" 3.87489
$4R_3^2 + 2R_o^2 =$	434.82..........	" 2.63831	
$4R_3^2 - 2R_o^2 =$	405.66..........		" 2.60816
$P_o(\theta) =$	24.75..........	" 1.39357	
$P_o(\rho) =$	18.48..........		" 1.26673

In the same way at each of the other sections the effect upon the bore of superposing the outer cylinders with the adopted shrinkages is first calculated, and thence the elastic strength of the assembled system is determined, the results being as shown by the curves in Figure 11.

63. Since the compression of the bore caused by superposing the hoop with the relative shrinkage ϕ_2 is by (38) $\frac{R_3^2 - R_2^2}{R_3^2 - R_o^2} E\phi_2$, the pressure at the surface of contact in the state of rest must be

$$\bar{P}_2 = \frac{R_2^2 - R_o^2}{2R_2^2} \frac{R_3^2 - R_2^2}{R_3^2 - R_o^2} E\phi_2 \tag{58}$$

and, since the whole compression of the bore in the state of rest is by (57) $\frac{R_2^2 - R_1^2}{R_2^2 - R_o^2} E\phi_1 + \frac{R_3^2 - R_2^2}{R_3^2 - R_o^2} E\phi_2$, we have similarly

$$\overline{P}_1 = \frac{R_1^2 - R_o^2}{2R_1^2}\left(\frac{R_2^2 - R_1^2}{R_2^2 - R_o^2} E\phi_1 + \frac{R_3^2 - R_2^2}{R_3^2 - R_o^2} E\phi_2\right) \qquad (59)$$

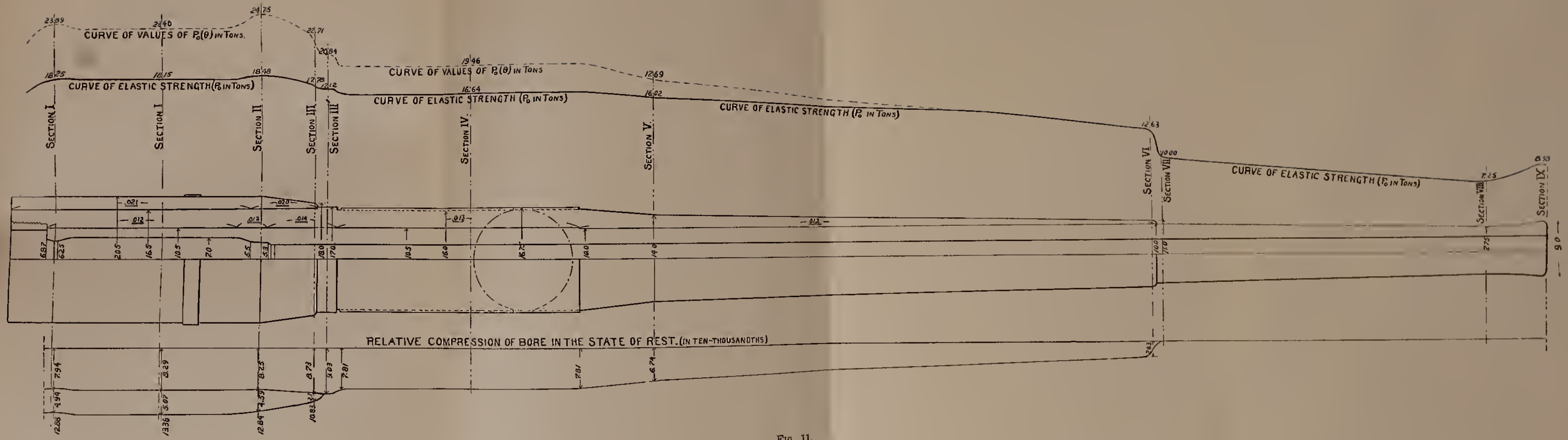
Thus we obtain the values of the pressures in the state of rest at each of the sections of the gun, and from them, together with the known value of P_o, the strains in the state of rest and of action may be found.

The following table gives the results of the calculations for the 5-inch gun shown in Figure 11:

SECTIONS.

	I′	*I*	*II*	*III*	*III′*	*IV*	*V*	*VI*	*VII*	*VIII*	*IX*
$P_o(\theta)$	23.89	23.40	24.75	22.71	20.84	19.46	17.69	12.63	10.00	7.25	8.98
P_o	18.25	18.15	18.48	17.78	17.12	16.64	16.02	12.63	10.00	7.25	8.98
$\overline{P}_2$	2.75	2.70	2.66	1.24	....	....	....	....	....	....	....
$\overline{P}_1$	5.41	4.83	6.14	5.45	4.54	3.93	3.29	1.28	....	....	....
$E\bar{e}_t(R_o)$	− 16.75	− 17.38	− 16.69	− 14.09	− 11.74	− 10.16	− 8.76	− 3.41	....	....	....
$E\bar{e}_t(R'_1)$	+ 5.32	+ 3.92	+ 7.59	+ 10.52	+ 11.65	+ 11.18	+ 11.22	+ 13.88	....	....	....
$E\bar{e}_t(R'_2)$	+ 13.79	+ 13.56	+ 13.34	+ 14.68	....	....	....	....	....	....	....
$Ee_t(R_o)$	+ 11.32	+ 11.61	+ 10.71	+ 12.60	+ 14.21	+ 15.63	+ 17.28	+ 20.00	+ 20.00	+ 20.00	+ 20.00
$Ee_t(R'_1)$	+ 16.07	+ 17.69	+ 15.51	+ 17.34	+ 18.40	+ 17.96	+ 18.90	+ 21.38	....	....	....
$Ee_t(R'_2)$	+ 18.88	+ 20.09	+ 17.10	+ 18.03	....	....	....	....	....	....	....

64. The method of procedure when there are more than three layers is exactly the same as has been explained for the cases of two and three layers respectively, and the formulæ already deduced are easily extended to cover any number of layers whatever. For the convenience of any one who may wish to use them, the formulæ for the case of four layers are given in full in an appendix.



CURVE OF VALUES OF $P_0(\theta)$ IN TONS.
CURVE OF VALUES OF $P_0(\theta)$ IN TONS
CURVE OF ELASTIC STRENGTH (P_0 IN TONS)
CURVE OF ELASTIC STRENGTH (P_0 IN TONS)
CURVE OF ELASTIC STRENGTH (P_0 IN TONS)
CURVE OF ELASTIC STRENGTH (P_0 IN TONS)
SECTION I
SECTION I
SECTION II
SECTION III
SECTION III
SECTION IV
SECTION V
SECTION VI
SECTION VII
SECTION VIII
SECTION IX
RELATIVE COMPRESSION OF BORE IN THE STATE OF REST. (IN TEN-THOUSANDTHS)

FIG. 11.

CHAPTER VII.

WIRE-WOUND GUNS.

65. A *wire-wound gun* differs from a built-up gun in that one or more of the outer layers of the latter are replaced in the former by steel wire, primarily for the purpose of increasing strength, steel in the form of wire having a higher elastic limit and tensile strength than it is practicable to obtain in large forgings. When first proposed, wire winding was relatively much more advantageous than it is to-day, when sound steel forgings of great size, and not greatly inferior in strength to steel wire, are readily procurable. Moreover, the promise of quicker and cheaper manufacture, often made for wire-wound systems of gun construction, has not as yet been fulfilled in practice. There is little doubt but that a wire-wound gun can be made of greater ultimate strength than a built-up gun of the same weight, or of equal strength, both elastic and ultimate, on less weight, but since the elastic strength after all depends upon the quality of the tube, or inner layer, which is the same in both systems, and since any reduction of weight by increasing the violence of recoil requires an increased weight for the gun mount and its supports, it is difficult to see any great advantage to be gained by substituting wire for solid forgings in gun construction. However, wire guns are in use, and possibly their use will become more extensive as experience in their manufacture increases, and so the principles of wire winding will be briefly discussed.

66. Winding with Constant Tension.—Let R_o and R_1 be the radii of the cylinder upon which the wire is to be wound, R_2 being the outer radius of the layers of wire, and let t_w be the constant tension of winding. Then, if Δr be the thickness of the wire, the application of a layer of wire at radius r will cause the radial pressure $p_r = t_w \dfrac{\Delta r}{r}$ at that radius.

But this pressure, by (24), will cause a circumferential true stress at the surface of the bore given by

$$\Delta E\bar{e}_t(R_o) = -\frac{2r^2 p_r}{r^2 - R_o^2} = -t_w \frac{2r\Delta r}{r^2 - R_o^2} \qquad (58)$$

And the total circumferential compression of the bore due to all the wire will be the sum of the partial compressions given by (58), which, since Δr is small, is given with practical exactness by

$$E\bar{e}_t(R_o) = -\int_{R_1}^{R_2} t_w \frac{2rdr}{r^2 - R_o^2} = -t_w \log_e\left(\frac{R_2^2 - R_o^2}{R_1^2 - R_o^2}\right) \tag{59}$$

But since the greatest elastic strength of the system will result from compressing the inner surface of the tube to its elastic limit, when in the state of rest, the proper tension of winding is given by putting $-\rho_o$ for $E\bar{e}_t(R_o)$ in (59), whence we have

$$t_w = \frac{\rho_o}{\log_e\left(\frac{R_2^2 - R_o^2}{R_1^2 - R_o^2}\right)} = \frac{0.4343\,\rho_o}{\log_{10}\left(\frac{R_2^2 - R_o^2}{R_1^2 - R_o^2}\right)} \tag{60}$$

Then the internal pressure which will bring the inner surface of the tube to its elastic limit of circumferential strain will be

$$P_o(\theta) = \frac{3\,(R_2^2 - R_o^2)}{4\,R_2^2 + 2R_o^2}(\theta_o + \rho_o) \tag{61}$$

and the internal pressure which will bring the inner surface of the tube to its elastic limit of radial strain will be

$$P_o(\rho) = \frac{2\,(R_2^2 - R_o^2)}{2\,R_2^2 - R_o^2}\rho_o \tag{62}$$

of which two values, when $\theta_o = \rho_o$, $P_o(\rho)$ will be the smaller and therefore the one to be accepted, if R_2^2 is greater than $\frac{5}{2}R_o^2$, which will practically always be the case.*

67. The compression at R_o in the state of rest, due to the wire, being ρ_o, the compression at R_1 will be $\frac{R_1^2 + 2R_o^2}{3R_1^2}\rho_o$, and so the true tension of the inner layer of wire in the state of rest will be $t_w - \frac{R_1^2 + 2R_o^2}{3R_1^2}\rho_o$, while that of the outer layer will, of course, be t_w.

In the state of action, P_o being the internal pressure, the true tension of the inner layer will be increased $\frac{2P_oR_o^2}{3(R_2^2 - R_o^2)}\left(1 + \frac{2R_2^2}{R_1^2}\right)$, while the true tension of the outer layer will be increased $\frac{2P_oR_o^2}{R_2^2 - R_o^2}$.

* If the compression of the bore is ρ (less than ρ_o), (61) and (62) will still give correct results provided ρ be put for ρ_o in (61) and $\frac{3\rho_o + \rho}{4}$ for ρ_o in (62).

If we suppose $P_o = P_o(\theta) = \frac{3(R_2^2 - R_o^2)}{4R_2^2 + 2R_o^2}(\theta_o + \rho_o)$, the true tension of the outer layer of wire in action will be $t_w + \frac{3R_o^2}{2R_2^2 + R_o^2}$ $(\theta_o + \rho_o)$, and this must not exceed the elastic limit of the wire.

68. As an example we will examine the case of a tube of radii $R_o = 5.0''$ and $R_1 = 8.0''$, with elastic limit $\theta_o = \rho_o = 18.0$ tons, with four inches of wire, for which $\theta = 40.0$ tons, wound upon it, the section of the wire being $0.2''$ wide by $0.1''$ thick.

$R_o = 5.0$ $R_o^2 = 25.0$	0.4343		log	9.63779
$R_1 = 8.0$ $R_1^2 = 64.0$	$\rho_o = 18.0$		"	1.25527
$R_2 = 12.0$ $R_2^2 = 144.0$	$R_2^2 - R_o^2 = 119.0$	log 2.07555		
	$R_1^2 - R_o^2 = 39.0$	" 1.59106	"	.89306
		.48449	"	9.68528
	$t_w = 16.14$ tons		"	1.20778
			"	3.35025
	$= 36143$ lbs.		"	4.55803

Therefore, the constant tension of winding which will compress the bore to its elastic limit is 36,143 pounds per square inch, or, the cross-section of the wire being 0.02 sq. in., 723 pounds on the wire.

$\theta_o = 18.0$	log 1.25527	log	1.25527
$R_2^2 - R_o^2 = 119.0$	" 2.07555	"	2.07555
3.0	" .47712		
2.0		"	.30103
	" 3.80794	"	3.63185
$2R_2^2 + R_o^2 = 313.0$	" 2.49554		
$2R_2^2 - R_o^2 = 263.0$		"	2.41996
$P_o(\theta) = 20.53$	" 1.31240		
$P_o(\rho) = 16.29$		"	1.21189

The least of these two values, $P_o(\rho) = 16.29$ tons, is the true elastic strength of the system.

$\rho_o = 18.0$	log	1.25527
$R_1^2 + 2R_o^2 = 114.0$	"	2.05690
	"	3.31217
$3R_1^2 = 192.0$	"	2.28330
10.69	"	1.02887

The compression at the outer surface of the tube, and the inner surface of the wire, due to the pressure of the wire in the state of rest, is 10.69 tons. Therefore the true tension of the inner layer of wire at rest is 16.14 — 10.69 = 5.45 tons per square inch.

$2P_o(\rho) =$	32.58	log 1.51295	log 1.51295
$R_o^2 =$	25.0	" 1.39794	" 1.39794
$R_1^2 + 2R_2^2 =$	352.0	" 2.54654	
		" 5.45743	
$3R_1^2 =$	192.0	" 2.28330	
		" 3.17413	" 2.91089
$R_2^2 - R_o^2 =$	119.0	" 2.07555	" 2.07555
	12.55	" 1.09858	
	6.84		" .83534

The increases of true tension at the inner and outer layers of wire caused by the internal pressure $P_o(\rho) = 16.29$ tons, are, respectively,

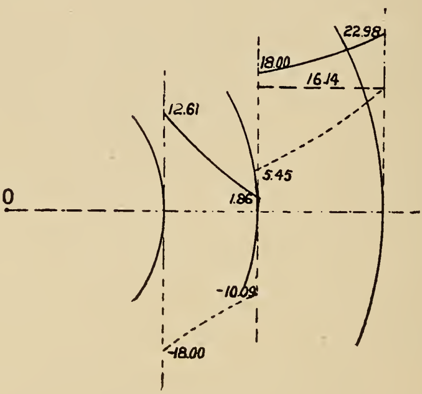


FIG. 12.

12.55 and 6.84 tons, so that the tensions of the inner and outer wires in the state of action are, respectively, 12.55 + 5.45 = 18.0 tons and 16.14 + 6.84 = 22.98 tons per square inch.

Under the internal pressure $P_o(\theta) = 20.53$ tons, the tension of the outer layer of wire would be 24.77 tons, which is still far within its elastic limit.

The circumferential true stresses for the case just discussed are graphically represented in Figure 12, in which the plus ordinates represent tensions and minus ordinates compressions. The dash

line shows the tension of winding, the dotted lines represent the state of rest, and the full lines the state of action with $P_o = 16.29$ tons.

69. When, as is usually the case, the number of layers of wire is such that a relatively small tension of winding compresses the tube to its elastic limit, and in the state of action the greatest strain is well within the elastic limit of the wire, a constant tension of winding serves every purpose, and, being easier of accomplishment than a varying tension, is naturally used. If, however, economy of material and weight is an object, it will be attained by winding the wire with a tension which varies from layer to layer in such a manner that in the state of action all the layers simultaneously reach the elastic limit of strain. The tension of winding which will bring about this result is determined as follows:

70. Let R_o and R_1 be the radii of the tube or cylinder upon which wire is wound to an outer radius R_2, and let T be the constant value of the circumferential true tension of each layer of wire in the state of action, so that $\frac{T}{E}$ is the circumferential strain throughout the mass of wire when P_o acts.

Then at any point r in the wire, the existing extension $\left(\frac{T}{E}\right)$ results from the concurrent action of three forces, namely, the tension of winding (t_w), the pressure of the outer layers of wire (p), and the internal pressure (P_o), and if we find the change of extension at r resulting from the removal of the outer layers of wire and the suppression of the internal pressure, and apply it to the extension $\frac{T}{E}$ which exists under those forces, the result will be the extension which was given to the wire in winding, and this multiplied by E is the desired tension of winding.

Let p and t be the radial pressure and circumferential tension at radius r under the internal pressure P_o. Then, by supposition, the circumferential strain being $\frac{T}{E}$, we have

$$\frac{1}{E}\left(t + \frac{p}{3}\right) = \frac{T}{E}$$

$$t = T - \frac{p}{3} \qquad (63)$$

But the product rp must always equal $\int_r^{R_2} t dr$, therefore

$$rp = \int_r^{R_2} \left(T - \frac{p}{3} \right) dr \tag{64}$$

from which, by differentiating, we get

$$rdp + pdr = - \left(T - \frac{p}{3} \right) dr$$

$$\frac{dp}{\frac{2}{3}p + T} = - \frac{dr}{r} \tag{65}$$

whence, by integration, knowing that, when $r = R_2$, $p = 0$ and $t = T$,

$$p = \frac{3T}{2} \left[\left(\frac{R_2}{r} \right)^{\frac{2}{3}} - 1 \right] \tag{66}$$

and this gives the value of the radial pressure at any point r within the wire in the state of action.

In accordance with the principle laid down in **49**, the change of circumferential strain at radius r, due to simultaneous changes (p_o and p) of the internal and external pressures, will, by (13), be given by

$$e_t = \frac{1}{E} \cdot \frac{6p_o R_o^2 - p(4R_o^2 + 2r^2)}{3(r^2 - R_o)} \tag{67}$$

Therefore, by putting in (67) $-P_o$ for p_o and $-\frac{3T}{2}\left[\left(\frac{R_2}{r} \right)^{\frac{2}{3}} - 1 \right]$ for p, we obtain the value of the change of circumferential strain at radius r due to the simultaneous removal of the wire beyond r and suppression of P_o, and $\frac{T}{E}$ plus this change of strain is the extension of the wire in winding, so that the tension of winding is given by

$$t_w = T + \frac{\frac{3T}{2}\left[\left(\frac{R_2}{r} \right)^{\frac{2}{3}} - 1 \right] (4R_o + 2r^2) - 6P_o R_o^2}{3(r^2 - R_o^2)}$$

$$t_w = \frac{T\left(\frac{R_2}{r} \right)^{\frac{2}{3}} (2R_o^2 + r^2) - R_o^2(3T + 2P_o)}{r^2 - R_o^2} \tag{68}$$

71. To determine the elastic strength of the cylinder when

wound with the varying tensions given by (68), since by (66) the external pressure on the tube in the state of action is

$$P_1 = \frac{3T}{2}\left[\left(\frac{R_2}{R_1}\right)^{\frac{2}{3}} - 1\right] \tag{69}$$

we have

$$P_o(\theta) = \frac{3(R_1^2 - R_o^2)\,\theta_o + 6P_1R_1^2}{4R_1^2 + 2R_o^2} \tag{70}$$

and

$$P_o(\rho) = \frac{3(R_1^2 - R_o^2)\,\rho_o + 2P_1R_1^2}{4R_1^2 - 2R_o^2} \tag{71}$$

of which two values of P_o the smaller should be taken.

72. Since (69) gives the value of the external pressure on the tube in the state of action, and since by (34) the change of pressure at R_1 due to the suppression of P_o is $-\dfrac{R_o^2\,(R_2^2 - R_1^2)}{R_1^2\,(R_2^2 - R_o^2)}\,P_o$, we have as the value of the pressure at R_1 in the state of rest:

$$\overline{P}_1 = \frac{3T}{2}\left[\left(\frac{R_2}{R_1}\right)^{\frac{2}{3}} - 1\right] - \frac{R_o^2\,(R_2^2 - R_1^2)}{R_1^2\,(R_2^2 - R_o^2)}\,P_o \tag{72}$$

and this must not exceed $\dfrac{R_1^2 - R_o^2}{2R_1^2}\,\rho_o$, or the bore will be compressed beyond its elastic limit of circumferential strain.

73. Since by (19) the circumferential true stress at R_1 caused by internal pressure P_o in a cylinder of outer radius R_2 is $\dfrac{2\,R_o^2\,(R_1^2 + 2\,R_2^2)}{3\,R_1^2\,(R_2^2 - R_o^2)}\,P_o$, the same expression gives the value of the change of stress caused by a change of internal pressure, and therefore the true tension of the inner layer of wire, which in the state of action is T, becomes in the state of rest, when P_o is suppressed,

$$E\bar{e}_t(R_1') = T - \frac{2R_o^2\,(R_1^2 + 2R_2^2)}{3R_1^2\,(R_2^2 - R_o^2)}\,P_o \tag{73}$$

while the tension of the outer layer of wire in the state of rest is, of course, the tension it was wound with, the value of which, found by putting $r = R_2$ in (68), is

$$E\bar{e}_t(R_2) = T - \frac{2P_oR_o^2}{R_2^2 - R_o^2} \tag{74}$$

74. As an example we will consider the case of the tube discussed in **67**, but with two inches of wire, instead of four inches, wound upon it:

$$\left.\begin{array}{lll} R_o = 5.0 & R_o^2 = 25.0 \\ R_1 = 8.0 & R_1^2 = 64.00 \\ R_2 = 10.0 & R_2^2 = 100.00 \end{array}\right\} \quad \left.\begin{array}{l} \theta_o = \rho_o = 18.0 \text{ tons} \\ \theta_1 = \rho_1 = 40.0 \quad \text{"} \end{array}\right\}$$

If the elastic strength of the wire, with its given number of layers, does not allow of compressing the tube to its elastic limit in the state of rest, we take the elastic limit of the wire (θ_1) for the value of T, but when, as in this and most other cases, there is surplus strength in the wire, it is necessary to find a value for T, less than θ_1, such that in the state of rest the tube is at its elastic limit of compression, as thus the greatest elastic strength is given to the system.

Putting $\frac{R_1^2 - R_o^2}{2 R_1^2}\rho_o$ for $\overline{P}_1$ in (72), we obtain the equation

$$P_1 = \frac{R_1^2 - R_o^2}{2R_1^2}\rho_o + \frac{R_o^2(R_2^2 - R_1^2)}{R_1^2(R_2^2 - R_o^2)}P_o$$

which, for this particular case, reduces to $P_1 = .1875\ P_o + 5.484$.

Either (70) or (71), according to which gives the smaller value of P_o, furnishes a second equation between P_1 and P_o, and from the two equations P_1 is found, and then, by (69), T. In this case an examination will show $P_o(\rho)$ to be smaller than $P_o(\theta)$, and (71), after substituting in it the values of R_o, R_1 and ρ_o, reduces to $P_1 = 1.6094\ P_o - 16.453$.

We thus find $P_1 = 8.377$ and $P_o(\rho) = 15.432$:

$$P_1 = 8.377 \quad \ldots\ldots\ldots\ldots\ldots \quad \log .92309$$

$$\frac{R_2}{R_1} = 1.25 \quad \ldots\ldots \log .09691$$

$$\left(\frac{R_2}{R_1}\right)^{\frac{2}{3}} = 1.1604 \ldots\ldots \tfrac{2}{3} \text{ " } .06461$$

$$\frac{3}{2}\left[\left(\frac{R_2}{R_1}\right)^{\frac{2}{3}} - 1\right] = .2406 \ldots\ldots\ldots\ldots\ldots \text{ " } 9.38130$$

$$T = 34.82 \quad \ldots\ldots\ldots\ldots\ldots \text{ " } \overline{1.54179}$$

The proper value for T can readily be found by trial instead of as just shown; thus, if we try $T = \theta_1 = 40.0$, we shall find the compression of the bore in the state of rest to be 21.6 tons, showing that T must be reduced about one-sixth (in order to reduce the

compression to 18.0 tons); then, after a second trial, a suitable value can be assigned to T.

We next find the tension of winding by (68), which in this case reduces to $t_w = \dfrac{\dfrac{8080.3}{r^{\frac{2}{3}}} + 161.61\, r^{\frac{4}{3}} - 3383.0}{r^2 - 25}$. Giving r in this equation the successive values 8.0, 8.5, 9.0, 9.5 and 10.0, we find, as the corresponding values of t_w, 31.25, 28.79, 26.96, 25.59 and 24.53. These are the tensions of winding in tons per square inch for the 1st, 5th, 10th, 15th and 20th or outer layer of wire, and, when reduced to pounds on the wire, become 1405, 1290, 1208, 1147 and 1099 pounds. The tensions for the other layers may either be calculated as these were or found by interpolation from them.

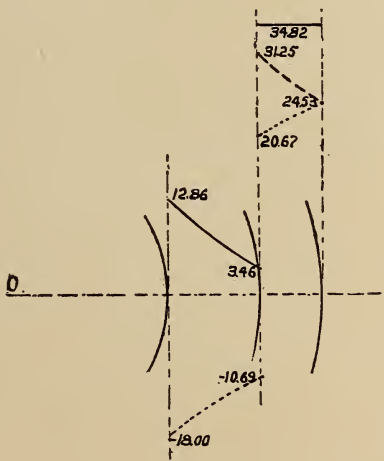


FIG. 13.

To find the tension of the inner wire in the state of rest we apply (73):

$2R_o^2 =$	50.0	..... log	1.69897
$R_1^2 + 2R_2^2 =$	264.0	..... "	2.42160
$P_o =$	15.432	..... "	1.18843
		"	5.30900
$3R_1^2 =$	192.0	..... "	2.28330
		"	3.02570
$R_2^2 - R_o^2 =$	75.0	..... "	1.87506
	14.146	..... "	1.15064

The true tension at R_1 being reduced 14.15 tons when P_o is suppressed, the true tension of the inner layer of wire in the state of rest is $34.82 - 14.15 = 20.67$ tons.

The circumferential true stresses for the case just discussed are represented in Figure 13, in which plus ordinates represent tensions and minus ordinates compressions. The dash line shows the tension of winding, the dotted lines the state of rest, and the full lines the state of action with $P_o = 15.43$ tons.

EXAMPLES VII.

(1) At what constant tension must 20 layers of steel wire 0.1″ thick be wound on a tube for which $R_0 = 5.0''$ and $R_1 = 8.0''$ to compress the bore to its elastic limit, 18 tons? Find $P_0(\theta)$ and $P_0(\rho)$ and the true tensions of the inner and outer wires both at rest and when $P_0(\rho)$ acts.

$t_w = 27.53$; $P_o(\theta) = 18.0$; $P_o(\rho) = 15.43$ tons.
16.84 and 27.53 tons, at rest.
30.98 and 37.82 tons, in action.

(2) A thickness of 2.5″ of wire is wound with the constant tension 32.5 tons per sq. in. on a tube for which $R_0 = 5.0''$ and $R_1 = 12.5''$. Find the compression of the bore in the state of rest, and the true tensions of the inner and outer wires both at rest and when $P_0(\rho)$ acts, if $\theta_0 = \rho_0 = 18.0$ tons.

$E\bar{e}_t(R_o) = 13.69$; $P_o(\rho) = 15.93$ tons.
26.48 and 32.50 tons, at rest.
31.63 and 36.48 tons, in action.

(3) A thickness of 1.25″ of wire is wound with the constant tension 15 tons per sq. in. on a tube for which $R_0 = 3.0''$, $R_1 = 5.25''$ and $\theta_0 = \rho_0 = 18$ tons. Find the compression of the bore in the state of rest, and the true tensions of the inner and outer wires both at rest and when $P_0(\rho)$ acts.

$E\bar{e}_t(R_o) = 8.75$; $P_o(\rho) = 13.82$ tons.
10.18 and 15.00 tons, at rest.
20.32 and 22.48 tons, in action.

(4) If, in the case of Example (2), the wire be so wound as to be under the constant true tension 34 tons per sq. in., find the values of $P_0(\theta)$, $P_0(\rho)$, and the compression of the bore and the

true tensions of the inner and outer wires in the state of rest. At what tension must the inner and the outer wires be wound?

$$P_o(\theta) = 19.65; \; P_o(\rho) = 15.91 \text{ tons.}$$
$$E\bar{e}_t(R_o) = 13.60; \; E\bar{e}_t(R_1') = 28.86 \text{ tons.}$$
$$t_w = 34.85 \text{ at } R_1; \; 30.02 \text{ at } R_2.$$

(5) Given $R_o = 4.0''$, $R_1 = 5.5''$, $R_2 = 8.0''$, $\theta_o = 30.0$, $\rho_o = 35.0$ and $T = 40.0$ tons (from R_1 to R_2 wire so wound as to have constant true tension of 40 tons when $P_o(\theta)$ acts); find $P_o(\theta)$, compression of bore at rest, true tension of inner and outer wires at rest, and tension of winding inner and outer wires.

$$P_o(\theta) = 28.39; \; E\bar{e}_t(R_o) = 31.29 \text{ tons.}$$
$$E\bar{e}_t(R_o) = 27.47; \; E\bar{e}_t(R_1') = 7.00 \text{ tons.}$$
$$t_w = 25.83 \text{ at } R_1; \; 21.07 \text{ at } R_2.$$

APPENDIX.

FORMULÆ FOR THE CASE OF COMPOUND CYLINDERS OF FOUR LAYERS.

$$
\left.
\begin{array}{ll}
(1) & P_3(\theta) = \dfrac{3(R_4^2 - R_3^2)\,\theta_3}{4R_4^2 + 2R_3^2} \\[2ex]
(2) & P_2(\theta) = \dfrac{3(R_3^2 - R_2^2)\,\theta_2 + 6P_3R_3^2}{4R_3^2 + 2R_2^2} \\[2ex]
(3) & P_1(\theta) = \dfrac{3(R_2^2 - R_1^2)\,\theta_1 + 6P_2R_2^2}{4R_2^2 + 2R_1^2} \\[2ex]
(4) & P_o(\theta) = \dfrac{3(R_1^2 - R_o^2)\,\theta_o + 6P_1R_1^2}{4R_1^2 + 2R_o^2} \\[2ex]
(5) & P_2(\rho) = \dfrac{3(R_3^2 - R_2^2)\,\rho_2 + 2P_3R_3^2}{4R_3^2 - 2R_2^2} \\[2ex]
(6) & P_1(\rho) = \dfrac{3(R_2^2 - R_1^2)\,\rho_1 + 2P_2R_2^2}{4R_2^2 - 2R_1^2} \\[2ex]
(7) & P_o(\rho) = \dfrac{3(R_1^2 - R_o^2)\,\rho_o + 2P_1R_1^2}{4R_1^2 - 2R_o^2} \\[2ex]
(8) & [P_o] = \dfrac{3(R_4^2 - R_o^2)(\theta_o + \rho_o)}{4R_4^2 + 2R_o^2}
\end{array}
\right\} \quad (A)
$$

If $P_o(\theta)$ is greater than $[P_o]$, the tube will be compressed beyond its elastic limit of compression (ρ_o) by shrinkages determined with the values of $P_3(\theta)$ $P_2(\theta)$, $P_1(\theta)$ and $P_o(\theta)$, and so the values of one or more of the assumed elastic limits θ_3, θ_2 and θ_1 must be reduced until $P_o(\theta)$ equals, or is less than, $[P_o]$.

$$
\left.
\begin{array}{ll}
(1) & S_3 = \dfrac{2R_3}{E}\left[\theta_3 + \dfrac{P_3(\theta)\,(4R_o^2 + 2R_3^2) - 6P_o(\theta)\,R_o^2}{3(R_3^2 - R_o^2)}\right] \\[2ex]
(2) & S_2 = \dfrac{2R_2}{E}\left[\theta_2 + \dfrac{P_2(\theta)\,(4R_o^2 + 2R_2^2) - 6P_o(\theta)\,R_o^2}{3(R_2^2 - R_o^2)}\right] \\[2ex]
(3) & S_1 = \dfrac{2R_1}{E}\left[\theta_1 + \dfrac{P_1(\theta)\,(4R_o^2 + 2R_1^2) - 6P_o(\theta)\,R_o^2}{3(R_1^2 - R_o^2)}\right]
\end{array}
\right\} \quad (B)
$$

In these expressions for the shrinkages, the values of θ_3, θ_2 and θ_1 are not necessarily the real elastic limits, but are the assumed

elastic limits with which the finally accepted values of $P_3(\theta)$, $P_2(\theta)$, $P_1(\theta)$ and $P_o(\theta)$ were calculated.

$$
\begin{aligned}
&(1)\quad \overline{P}_3 = P_3(\theta) - \frac{R_o^2\,(R_4^2 - R_3^2)}{R_3^2\,(R_4^2 - R_o^2)}\,P_o(\theta)\\
&(2)\quad \overline{P}_2 = P_2(\theta) - \frac{R_o^2\,(R_4^2 - R_2^2)}{R_2^2\,(R_4^2 - R_o^2)}\,P_o(\theta) \qquad \text{(C)}\\
&(3)\quad \overline{P}_1 = P_1(\theta) - \frac{R_o^2\,(R_4^2 - R_1^2)}{R_1^2\,(R_4^2 - R_o^2)}\,P_o(\theta)
\end{aligned}
$$

These are the pressures at the surfaces of contact *in the state of rest,* $P_3(\theta)$, $P_2(\theta)$, $P_1(\theta)$ and $P_o(\theta)$ being the values of the pressures in the state of action used in calculating the assigned shrinkages.

$$
\bar{e}_t(R_o) = -\frac{R_2^2 - R_1^2}{R_2^2 - R_o^2}\,\frac{S_1}{2R_1} - \frac{R_3^2 - R_2^2}{R_3^2 - R_o^2}\,\frac{S_2}{2R_2} - \frac{R_4^2 - R_3^2}{R_4^2 - R_o^2}\,\frac{S_3}{2R_3} \qquad \text{(D)}
$$

This is the circumferential strain at the surface of the bore caused by the superposition of the three outer layers with their respective shrinkages S_1, S_2 and S_3, the successive terms being the three circumferential strains produced by the three successive layers. $2R_o\bar{e}_t(R_o)$ is the change of diameter (contraction) of the bore from its free state to that of complete assemblage of the system, and $-E\bar{e}_t(R_o)$ is the circumferential compression of the bore in the state of rest.

The radial strain at the surface of the bore in the state of rest is $\bar{e}_p(R_o) = -\frac{1}{3}\,\bar{e}_t(R_o)$, so that it is under a true tension radially one-third as great as its circumferential compression. Therefore the real elastic strength of the system when assembled with shrinkages S_1, S_2 and S_3 is the least of the two following values of P_o:

$$
\begin{aligned}
&(1)\quad P_o^{(1)} = \frac{3(R_4^2 - R_o^2)}{4R_4^2 + 2R_o^2}\,(\theta_o - E\bar{e}_t(R_o))\\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(E)}\\
&(2)\quad P_o^{(2)} = \frac{3(R_4^2 - R_o^2)}{4R_4^2 - 2R_o^2}\,(\rho_o - \tfrac{1}{3}\,E\bar{e}_t(R_o))
\end{aligned}
$$

In these expressions it is important to note that $\bar{e}_t(R_o)$ is a *negative strain,* so that the last factor in each of the two values of P_o is *numerically the sum,* not the difference, of the elastic limits (of tension and compression respectively) and the true stresses at the surface of the bore in the state of rest (circumferential and radial respectively).

The pressures in the state of rest may be computed directly from the shrinkages by the following formulæ:

$$\begin{aligned}
&(1)\quad \overline{P}_3 = E\frac{R_3^2-R_o^2}{2R_3^2}\cdot\frac{R_4^2-R_3^2}{R_4^2-R_o^2}\cdot\frac{S_3}{2R_3}\\
&(2)\quad \overline{P}_2 = E\frac{R_2^2-R_o^2}{2R_2^2}\left(\frac{R_3^2-R_2^2}{R_3^2-R_o^2}\frac{S_2}{2R_2}+\frac{R_4^2-R_3^2}{R_4^2-R_o^2}\frac{S_3}{2R_3}\right)\\
&(3)\quad \overline{P}_1 = E\frac{R_1^2-R_o^2}{2R_1^2}\left(\frac{R_2^2-R_1^2}{R_2^2-R_o^2}\frac{S_1}{2R_1}+\frac{R_3^2-R_2^2}{R_3^2-R_o^2}\frac{S_2}{2R_2}\right.\\
&\qquad\qquad\qquad\qquad\left.+\frac{R_4^2-R_3^2}{R_4^2-R_o^2}\frac{S_3}{2R_3}\right)
\end{aligned}\qquad \text{(F)}$$

The terms in the parentheses are the values of the circumferential strains at R_o caused by the assemblage of the successive layers, their sum with the negative sign being the total compressive strain at the surface of the bore as given by equation (D).

From the pressures in the state of rest, as given by (F), the pressures in the state of action may be found by equations (C), and the true circumferential tension of the inner surface of any layer can then be found by

$$Ee_t(R'_{n-1}) = \frac{P_{n-1}(4R_n^2+2R_{n-1}^2)-6P_nR_n^2}{3(R_n^2-R_{n-1}^2)} \qquad \text{(G)}$$

In this R_n and R_{n-1} are the outer and inner radii of any layer, P_n and P_{n-1} are the outer and inner pressures (either of action or of rest) and Ee_t (R'_{n-1}) is the true circumferential tension at the inner surface of the layer resulting from the action of P_n and P_{n-1}.

Similarly, the true radial compression at the inner surface of any layer, either in the state of rest or of action, is given by

$$Ee_p(R'_{n-1}) = \frac{P_{n-1}(4R_n^2-2R_{n-1}^2)-2P_nR_n^2}{3(R_n^2-R_{n-1}^2)} \qquad \text{(H)}$$